Almost Homogeneous Projective 3-Folds

Dissertation

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Contents

CHAPTER 1

Introduction

If X is a projective variety with at most terminal singularities and G is a connected linear algebraic group which acts algebraically and has an open orbit, then we say that X is "almost homogeneous" with respect to G.

In the case $\dim X = 2$, all possibilities for X and G are known. The minimal models are the HIRZEBRUCH surfaces, and all other almost homogeneous surfaces can be obtained from these by sequences of blowing up G-fixed points.

In dimension 3, the birational geometry is considerably more difficult. Take for example the group $(\mathbb{C}^3, +)$ acting on $\mathbb{P}_3$ by affine translation. Given a curve C contained in the G-fixed divisor at infinity, we can construct a G-almost homogeneous variety X_C equipped with an equivariant birational morphism $X_C \to \mathbb{P}_3$ by blowing up C and then equivariantly resolving the singularities. This yields uncountably many equivariant birational models of $(\mathbb{P}_3, (\mathbb{C}^3, +))$, which leads to difficulties in any classification theory.

To overcome this problem we propose to apply MORI theory in order to determining a reasonable class of minimal almost homogeneous varieties generating the set of all almost homogeneous varieties by sequences of well-understood transformations.

Starting with a smooth threefold X, MORI theory gives us an equivariant map $\phi : X \to Y$ call the "contraction of an extremal ray". If $\dim Y < 3$, we call X minimal. If $\dim Y = 1, 2$, then we show that except for one special variety, which can be completely described, X is either a $\mathbb{P}_1$-bundle over an almost homogeneous rational surface, or X is a linear $\mathbb{P}_2$-bundle over $\mathbb{P}_1$.

If Y is a point, then X is FANO and has PICARD-number $\rho(X) = 1$, so that we can look it up in ISKOVSKIH's list. The case that Y is of dimension 3 is more difficult, because Y will generally have terminal singularities. MORI theory allows us to contract again, but we have to include varieties with terminal singularities into our considerations. The result in the case that $\dim Y = 1$ or 2 still holds, but there might be new singular varieties where Y is a point.

One might hope that there exists a sequence of MORI contractions

$$X \xrightarrow{\phi^{(1)}} X^{(1)} \xrightarrow{\phi^{(2)}} \dots \xrightarrow{\phi^{(n)}} X^{(n)} \xrightarrow{\phi^{(n+1)}} Y \tag{1.1}$$

where $\dim Y < 3$, i.e. $X^{(n)}$ would be the desired minimal model. However, such sequences may terminate with $\dim Y = 3$ and Y having complicated singularities. In the usual MORI theory one would apply at this point a sequence of very difficult birational transformations, called "flips", before continuing with contraction.

One of the key points of this paper is that we can use the G-action in order to direct the steps of the MORI minimal model program. After resolving the singularities of X and then blowing up, we have a sequence of contractions as in (1.1) with

$\phi^{(1)}, \dots, \phi^{(n)}$ being simple blow-downs with smooth centers. In particular, no flips occur.

Our results can be summarized as follows: Let X be a smooth projective variety of dimension 3 which is almost homogeneous with respect to the algebraic action of a linear algebraic group G. Then either $G \cong SL_2$, and X is a compactification of SL_2/Γ, where $\Gamma < SL_2$ is a finite subgroup, or after equivariantly resolving the singularities of X, a sequence of blowing up followed by a sequence of blowing down, we obtain a variety X' which is one of the following:

1. $\mathbb{P}_3$
2. Q_3, the 3-dimensional quadric
3. a linear $\mathbb{P}_1$-bundle over a surface
4. a linear $\mathbb{P}_2$-bundle over $\mathbb{P}_1$.

If G is solvable in case 3, then we may even take X' to be the compactification of a line bundle. The cases (1) and (2) correspond to Y being a point in (1.1): both $\mathbb{P}_3$ and Q_3 allow a Mori contraction to a point. The cases (3) and (4) correspond to $\dim Y = 1$ or 2. Here the Mori contractions are just the bundle maps.

We have chosen not to discuss the case that X is an equivariant compactification of SL_2/Γ because a combinatorial classification exists ([**MJ87**]).

The following is an outline of this paper:

Part 1. In chapter 2 we recall known results from the theory of almost homogeneous spaces and from the Mori minimal model program for varieties of dimension 3.

Chapter 3 is then devoted to questions concerning equivariance of maps. It is shown that all the mappings which are encountered (extremal contractions, flips, resolutions, blow-ups of G-stable subsets, etc) are equivariant. At the end we show that almost homogeneity implies that there always is a Mori-contraction, so that the minimal model program terminates with a contraction to a variety of lower dimension.

Part 2. Here the structure of those varieties is determined which admit a Mori contraction to a variety of dimension one or two. Throughout this part, X is always the minimal model (in the above sense) and $\phi : X \to Y$ ($\dim Y < 3$) the contraction.

The case where Y is a curve is dealt with in chapter 4. After excluding all the other possibilities, two cases are left. First, there is a very special quadric bundle which will be described in example 4.12 and secondly, there are $\mathbb{P}_2$-bundles over $\mathbb{P}_1$. All such bundles can indeed occur.

The final case handled in chapter 5: Y is a surface. Here it is shown that Y is an almost homogeneous surface and X as well as Y are smooth. It will turn out that X is a linear $\mathbb{P}_1$-bundle over Y. In case that X is not the compactification of a line bundle, it will be possible in many cases to transform X into one, using only composites of equivariant blow-ups with smooth center.

Part 3. Naturally, the question arises in which way almost homogeneous threefolds are linked to their minimal models. Of course, there are the various steps in the minimal model program. While the theory for contractions of smooth varieties is well-developed, contractions and flips in the singular case create significant further difficulties. It might be possible to make use of the classification [**KM92**], but again this is an extremely involved matter and thus we have chosen another approach.

In chapter 6 the group action is used to find equivariant rational fibrations of the almost homogeneous 3-dimensional varieties. These results will be used in order to show that in all relevant cases we may assume that X has a minimal model M with bundle structure.

Our basic idea is now carried out in chapter 7: we show that by equivariantly blowing up and down we can adjust the geometry of X and M so that the regularized map between them factors into a sequence of blow-downs. As a net result, we show the entire minimal model program can be carried out in our context by only equivariantly blowing up and down.

Part 1

Basics

CHAPTER 2

Preparations

This chapter contains no new results. Everything presented here is either implicitly or explicitly in the literature.

1. Almost Homogeneous Varieties

DEFINITION 2.1. Let X denote an irreducible normal algebraic variety. If there exists a connected linear algebraic LIE group G acting on X such that the associated map $G \times X \to X$ is algebraic, one speaks of an "algebraic group action of G on X".

REMARK 2.2. A group action being algebraic implies that

1. If $C \subset X$ is an algebraic subvariety, and $I < G$ an algebraic subgroup, then $I.C$ contains a set which is ZARISKI open in its closure. This is a application of the constructibility of images of algebraic maps (see e.g. [**Hum75**, p. 23]).
2. If I is an algebraic subgroup of G, then I has finitely many components. In particular, if $\dim I = 0$, I is finite.

DEFINITION 2.3. If the group G has an open orbit in X, then X is called "almost homogeneous" (with respect to the group action of G), or we say that G "acts almost transitively". In this case let Ω denote the union of all open orbits. If $\Omega \neq X$, X is called "strictly almost homogeneous".

2. Properties of Almost Homogeneous Varieties

LEMMA 2.4. *Every G-orbit is* ZARISKI *open in its closure.*

PROOF. Since G acts algebraically, the associated map $\rho : G \times X \to X$ is algebraic. Given any point $x \in X$, the set $G \times \{x\}$ is algebraic. Now CHEVALLEY's theorem applies, hence the image $\rho(G \times \{x\}) = G.x$ is constructible. In particular, $G.x$ contains a maximal ZARISKI open subset U. If there existed point $y \in G.x \setminus U$, we also had a $g \in G$ such that $g.y \in U$. In particular, $U \cap g^{-1}(U)$ is open, contained in $G.x$ and contains y. So U was not maximal! Hence $G.x$ is ZARISKI open in its closure. □

COROLLARY 2.5. *The "exceptional set" $E := X \setminus \Omega$ is an algebraic subvariety of X. In particular, since X is irreducible implies that Ω is connected and is the unique open orbit.*

PROPOSITION 2.6. *Let X be a smooth projective variety, almost homogeneous under the action of a linear algebraic group. Then the first* BETTI *number is zero: $b_1(X) = 0$.*

PROOF. The dimension of the ALBANESE torus $\mathrm{Alb}(X)$ associated to X is is the half of $b_1(X)$. The map $X \to \mathrm{Alb}(X)$ is equivariant and yields a surjective algebraic group morphism: $G \to \mathrm{Aut}(Alb(X))^0 \cong Alb(X)$. Since G is linear algebraic, $\mathrm{Alb}(X)$ is trivial. Hence $b_1(X) = 0$. □

COROLLARY 2.7. *Let X be a projective variety which is almost homogeneous under the action of a linear algebraic group. If $\dim(X) = 2$, then X is rational.*

PROOF. It is sufficient to show this corollary in the case that X is smooth — or else be desingularize equivariantly, if necessary. The fibers of the ALBANESE map $X \to \mathrm{Alb}(X)$ are unirational. If $\dim(X) \leq 2$, then unirationality implies rationality. □

PROPOSITION 2.8. *Let X be a compact projective algebraic variety with $b_1(X) = 0$, and let L be a basepoint-free line bundle on X. Let G be as above, acting on X. Then the induced morphism $X \to \mathbb{P}_n$ is equivariant.*

PROOF. See [**HO80**, p. 18] or [**Bla56**]. We need the connectedness of G here. □

COROLLARY 2.9. *Let X and G as above. Then X is equivariantly embedded into a $\mathbb{P}_n$.*

The following fixed point theorem of BOREL will be used at numerous points in the sequel.

PROPOSITION 2.10. *Suppose that G is a connected solvable* LIE *group in $\mathrm{Aut}(\mathbb{P}_n)$ which stabilizes a compact projective algebraic variety X. Then G has a fixed point in X.*

PROOF. See [**HO80**, p. 32] □

LEMMA 2.11. *Let X be a minimal smooth algebraic surface which is almost homogeneous with respect to an algebraic group action. Then X is either $\mathbb{P}_2$ or a* HIRZEBRUCH *surface $\Sigma_n, n \neq 1$, or a ruled surface over a torus.*

PROOF. By almost homogeneity, we can always find $\dim X$ elements $v_1, \ldots, v_n$ of the LIE-algebra $Lie(G)$ such that the associated fundamental vector fields $\tilde{v_1}, \ldots, \tilde{v_n}$ are linearly independent at generic points of X. In other words

$$\sigma := \tilde{v_1} \wedge \ldots \wedge \tilde{v_n}$$

is a non-trivial holomorphic section of the anticanonical bundle $-K_X$. Hence the KODAIRA-dimension of X is $\kappa(X) = -\infty$. Looking at the ENRIQUES classification of surfaces and taking into account that X is projective, it follows that X is either

1. minimal rational, i.e. $\mathbb{P}_2$ or a HIRZEBRUCH surface, or
2. X is a ruled surface of genus ≥ 1.

In the second case, we use the fact that a regular map with connected fibers between two normal compact varieties is always equivariant. Hence if $\phi : X \to Y$ is the ruling, then Y has to be an almost-homogeneous curve of genus ≥ 1. There is only one possibility, namely that Y is a torus. □

COROLLARY 2.12. *Let X be a smooth surface which is almost homogeneous with respect to an algebraic group action of a linear algebraic group. Then X is either $\mathbb{P}_2$ or a* HIRZEBRUCH *surface Σ_n, possibly blown up at finitely many points.*

PROOF. Let X' be a minimal model of X which is obtained from X by blowing down finitely many exceptional curves of first type. Again, all the blow-downs are equivariant. By proposition 2.6, $\mathrm{Alb}(X) = 0$. □

3. Singularities

DEFINITION 2.13. Let $H \subset X$ be a divisor. If X admits a covering $(U_\alpha)_{\alpha \in A}$ with holomorphic functions $f_\alpha : U_\alpha \to \mathbb{C}$ such that $H \cap U_\alpha = \{f_\alpha = 0\}$ and $\forall \alpha, \beta \in A : \frac{f_\alpha}{f_\beta} \in \mathcal{O}^*_{U_\alpha \cap U_\beta}$, then H is called "CARTIER". If there exists $n \in \mathbb{N}$, so that nH is CARTIER, then H is called $\mathbb{Q}$-CARTIER.

If every divisor on X is $\mathbb{Q}$-CARTIER, X is called $\mathbb{Q}$-factorial.

DEFINITION 2.14. Let S denote the set of singular points of X and $K_{X \setminus S}$ the canonical bundle on the smooth points. The line bundle $K_{X \setminus S}$ is associated to a divisor which can be extended to X. One refers to the extended divisor as K_X. If K_X is CARTIER, then X is called "GORENSTEIN". If K_X is $\mathbb{Q}$-CARTIER, then X is said to be "$\mathbb{Q}$-GORENSTEIN".

The number

$$r := \min\{n \in \mathbb{N} : nK_X \text{ is CARTIER}\}$$

is called the "index of the variety X".

REMARK 2.15. Let X be a normal $\mathbb{Q}$-GORENSTEIN variety, r the index of X and $\pi : Y \to X$ an arbitrary resolution of the singularities. Then

$$rK_Y = \pi^*(rK_X) + \sum_i ra_i E_i,$$

where $a_i \in \mathbb{Q}$ and E_i are π-exceptional divisors.

DEFINITION 2.16. The number a_i is called the "discrepancy along the divisor E_i", $a_{(\pi,Y)} := \min_i\{a_i\}$ the "discrepancy of the resolution Y of X", and

$$discrep(X) := \inf\{a_{(\pi,Y)} \mid \pi : Y \to X \text{ is a resolution}\} \in \mathbb{R} \cup \{-\infty\}$$

is the "discrepancy of X".

The well-defined number $\mathrm{diff}(X) := \#\{i : a(\pi, Y) < 1\}$ is called the "difficulty of X".

LEMMA 2.17. *The difficulty is well-defined, i.e. it is independent of the resolution.*

PROPOSITION 2.18. *Either $discrep(X) \in [-1, 1]$ or $discrep(X) = -\infty$.*

PROOF. The proof can be found in [**CKM88**, p. 39]. □

TERMINOLOGY 2.19. The $\mathbb{Q}$-factorial singularities are divided into the following classes, depending on $discrep(X)$

$discrep(X)$	name
≥ -1	X has log-canonical singularities.
> -1	X has log-terminal singularities.
≥ 0	X has canonical singularities.
> 0	X has terminal singularities.

If A is a subvariety of X and furthermore $\forall i : \pi(E_i) \not\subseteq A \vee a_i > 0$, one says "$X$ has terminal singularities along A".

REMARK 2.20. If X is a variety having at most terminal singularities and $\tilde{X}$ is a blow-up of X, then $\tilde{X}$ itself has at most terminal singularities only.

LEMMA 2.21. *Let X be a variety and $\pi : Y \to X$ a resolution of the singularities such that $a_{(\pi,Y)} = c > 0$. Then X has terminal singularities.*

PROOF. Let $\pi' : Y' \to X$ be another resolution of singularities. Then by HIRONAKA [**Hir62**] there is a commutative diagram of projective varieties and birational maps between them. We may furthermore assume that ϕ is a combination of blow-ups and that all mappings are isomorphic outside of preimages of singularities.

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow^{\phi} & & \searrow^{\phi'} & \\ Y & & & & Y' \\ & \searrow_{\pi} & & \swarrow_{\pi'} & \\ & & X & & \end{array}$$

So every π or $\phi \circ \pi$-exceptional divisor is already π' or $\phi' \circ \pi'$-exceptional. Thus:

$$\begin{aligned} K_Y &= \pi^*(K_X) + cE_0 + \sum_{i>0} a_i E_i \\ K_{Y'} &= {\pi'}^*(K_X) + \sum_i a_i' E_i' \\ K_{\tilde{Y}} &= \phi^*(K_Y) + \sum_j b_i D_i \\ &= \phi^*\pi^* K_X + c\phi^* E_0 + \sum_{i>0} a_i \phi^* E_i + \sum_j b_i D_i \\ &= {\phi'}^*{\pi'}^* K_X + \sum_i a_i' {\phi'}^* E_i' + \sum_j b_i' D_i' \end{aligned}$$

However, since Y and $\tilde{Y}$ are already smooth, $b_i \geq 1$ and all discrepancies along $\phi \circ \pi$-exceptional divisors are already $\geq c$. Comparing the coefficients we have the same property for the resolution $\phi' \circ \pi'$. But the a_i' are also the coefficients of the strict transforms of E_i' under ${\phi'}^{-1}$. So the claim is valid for the resolution π' as well. □

The terminal singularities of 3-dimensional varieties have been completely classified by MORI. See [**Rei87**] for a survey. In particular

PROPOSITION 2.22. *Terminal singularities of three dimensional varieties are isolated points.*

We will need the following lemmas in the next chapters.

LEMMA 2.23. *Let X be a variety with isolated singularities (e.g. terminal and $\dim X = 3$). Then* BERTINI*'s theorem holds, i.e. a generic element of a basepoint-free linear system is smooth.*

PROOF. Let L be a basepoint-free line bundle on X. Let $\pi : \tilde{X} \to X$ be a resolution of singularities. Then $\pi^*(L)$ is again basepoint-free. $\pi_* : |\pi^*(L)| \to |L|$ is an isomorphism of linear systems. Hence a generic element of $|L|$ is generic in

$|\pi^*(L)|$ and vice versa. In particular, a generic element of $|\pi^*(L)|$ is smooth and its π-image is disjoint from the set of singularities. Outside of the (preimages of) singularities, π is an isomorphism. Hence the image of a generic element is still smooth.

□

LEMMA 2.24. *Let X be a variety with terminal singularities. Let L be a basepoint-free line bundle. Then the adjunction formula holds for the generic element in the linear system $|L|$.*

PROOF. Using the notation of the proof for lemma 2.23 and applying the adjunction formula on $\tilde{X}$, the claim follows from the fact that the canonical bundles of K_X and $K_{\tilde{X}}$ agree outside of (preimages of) the singularities. □

4. Mori Theory

4.1. Basic Notations.

NOTATION 2.25. Let X be a projective, normal $\mathbb{Q}$-factorial variety of arbitrary dimension. For $C \subset X$ a curve, let $[C]$ be the class of C in $H_2(X, \mathbb{R})$. Define

$$\mathrm{N}(X) := \{\sum_i a_i[C_i] : C_i \text{ is a curve in } X, a_i \in \mathbb{R}\} \subseteq H_2(X, \mathbb{R})$$

and

$$\mathrm{NE}(X) := \{\sum_i a_i[C_i] : C_i \text{ is a curve in } X, a_i \in \mathbb{R}^+\} \subseteq \mathrm{N}(X) \ .$$

NOTATION 2.26. $\mathrm{N}(X)$ is a finite dimensional $\mathbb{R}$-vector subspace of $H_2(X, \mathbb{R})$ with the usual topology. Let $\overline{NE(X)}$ be the closure of $\mathrm{NE}(X)$ with respect to this topology. Then $\overline{NE(X)}$ is a convex $\mathbb{R}^+$-cone.

DEFINITION 2.27. Let K be an arbitrary $\mathbb{R}^+$-cone. A subcone $C \subset K$ is said to be "extremal" if the following condition is fulfilled for all $a, b \in K$: If there exists a $\lambda \in [0, 1]$ such that $\lambda a + (1 - \lambda)b \in C$, then $a \in C$ or $b \in C$.

The next lemma deals with simple properties of convex cones.

LEMMA 2.28. *Let $\phi : V \to V'$ denote a linear surjective map between finite dimensional $\mathbb{R}$ vector spaces. Let $K \subset V$ and $K' \subset V'$ be closed convex $\mathbb{R}^+$ cones such that $\phi(K) = K'$. Let $C \subset K$ and $C' \subset K'$ be closed convex subcones such that $C = \phi^{-1}(C')$. Then C is extremal if and only if C' is.*

PROOF. To prove C extremal $\Longrightarrow C'$ extremal, let $a', b' \in K'$ and $\lambda \in [0, 1]$ such that $\lambda a' + (1 - \lambda)b' \in C'$. Take some $a \in \phi^{-1}(a')$ and $b \in \phi^{-1}(b')$. Clearly we have $\lambda a + (1 - \lambda)b \in C$. Since C is extremal we may assume that $a \in C$. So $\phi(a) = a' \in \phi(C) = C'$. Thus C' is extremal.

The other direction is similar. Assume C' to be extremal. Let $a, b \in K$ and $\lambda \in [0, 1]$ such that $\lambda a + (1 - \lambda)b \in C$. Then $\lambda\phi(a) + (1 - \lambda)\phi(b) \in C'$ and, without loss of the generality, $\phi(a) \in C'$. Thus $a \in C$ and C is extremal. □

NOTATION 2.29. Let H be a divisor on X. Define

$$(H)_{\leq 0} := \{[C] \in H_2(X, \mathbb{R}) : H.C \leq 0\}.$$

Use the symbols $(H)_{=0}$, ... analogously.

Certain essential ingredients of MORI theory are based on the investigations of KLEIMAN on the cone $\mathrm{NE}(X)$, in particular his criterion for the ampleness of $\mathbb{Q}$-CARTIER divisors. A proof can be found in [**Kle66**, Chapter IV, §4, Theorem 1].

THEOREM 2.30 (KLEIMAN's criterion). *Let X be a normal compact algebraic variety. A $\mathbb{Q}$-CARTIER divisor is ample if and only if $\overline{NE(X)} \setminus \{0\} \subseteq (H)_{>0}$.*

4.2. Extremal Contractions. We begin with the main theorem that provides the existence and a description of the extremal contractions. A proof can be found in [**Mor82**] or [**CKM88**].

THEOREM 2.31 (cone- and contraction theorem). *Let X be a projective, normal, $\mathbb{Q}$-factorial variety of arbitrary dimension having at worst canonical singularities. Then there is a family $(C_i)_{i\in I}$ of curves such that:*

$$\overline{NE(X)} = \overline{NE(X)} \cap (K_X)_{\geq 0} + \sum_{i\in I} \mathbb{R}^+[C_i]$$

If $H \subset X$ is an ample $\mathbb{Q}$-divisor, then there exists $I_H \subseteq I$, such that:

$$\overline{NE(X)} \cap (K_X + H)_{\leq 0} = (K_X + H)_{=0} \cap \overline{NE(X)} + \sum_{i\in I_H} \mathbb{R}^+[C_i].$$

This decomposition has the following properties:

1. *The family $(C_i)_{i\in I}$ is minimal.*
2. *Every ray $\mathbb{R}^+[C_i]$ is extremal as a closed convex subcone of $\overline{NE(X)}$.*
3. *The set $I_H \subseteq I$ is finite.*
4. *If X is smooth, C_i can be chosen to be rational curves. Furthermore $0 \geq K_X.C_i \geq -\dim(X) - 1$.*
5. *For all C_i there is a basepoint free line bundle F_i giving a morphism $\phi_i : X \to X_i$ to a projective, normal variety X_i. A curve C is mapped to a point if and only if $[C] \in \mathbb{R}^+[C_i]$.*

Figure 1 shows a typical example of the cone $\overline{NE(X)}$. On the side where $(K_X)_{>0}$ one knows very little about the extremal rays, e.g. the cone might even be round. On the side $(K_X)_{<0}$ a transversal section through $\overline{NE(X)}$ is almost polygonal. Note, however, that the extremal rays (i.e. the edges) might accumulate at the hyperplane $(K_X)_{=0}$.

TERMINOLOGY 2.32. Those rays in $\overline{NE(X)}$ that are extremal in the convex-theoretic sense are called "extremal" . The classes of the curves C_i are extremal in this sense. Note that many authors use "extremal" only for those rays that are convex-extremal and lie in the half-space $(K_X)_{<0}$.

The curves C_i are called "extremal curves" . The associated mapping ϕ_i is called an "extremal contraction", a "MORI contraction" or a "contraction of the curve C_i" .

LEMMA 2.33. *Suppose that $\phi : X \to Y$ is an extremal contraction. Let C be an element of the associated extremal ray. If $D \in \mathrm{Pic}(X)$ is a bundle with $C.D > 0$ and $H \in \mathrm{Pic}(Y)$ is ample, then $D + k\phi^*(H) \in \mathrm{Pic}(X)$ is ample for $k >> 0$.*

PROOF. This is a corollary of KLEIMAN's ampleness criterion (theorem 2.30) and the construction of the extremal contraction as the morphism associated to $\phi^*(H)$. □

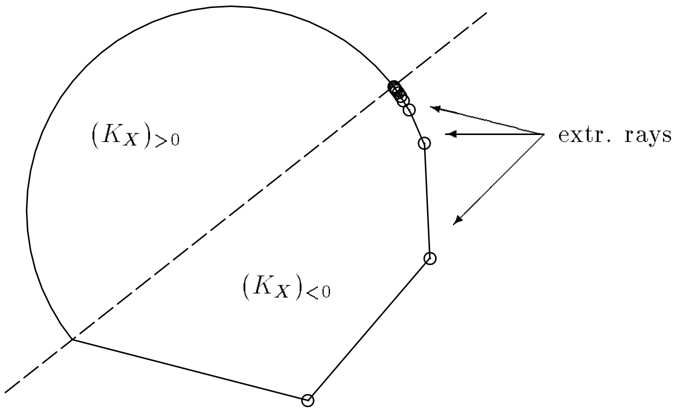


FIGURE 1. transversal section through $\overline{NE(X)}$

NOTATION 2.34. Let ϕ be a morphism between algebraic varieties $\phi : X \to Y$. Set

$$A_\phi := \{x \in X : \phi \text{ is not isomorphic in any neighborhood of } x\ \}$$

THEOREM 2.35. *Let X be a projective variety with at worst canonical singularities and $\phi : X \to Z$ an extremal contraction. Then the equality of* PICARD *numbers holds $\rho(X) = \rho(Z) + 1$ and $-K_X$ is ϕ-ample. In particular, for all $z \in Z : -K_X|_{\phi^{-1}(z)}$ is ample on $\phi^{-1}(z)$. Furthermore, Z is normal and ϕ is one of the following:*

Fibration: *(*$\operatorname{codim} A_\phi = 0$*) Then* $\dim Z < \dim X$ *and the generic fiber is a* FANO *variety. Z is $\mathbb{Q}$-factorial. If X contains only canonical (respectively terminal) singularities the singularities of Z are canonical (respectively terminal) as well.*

Divisorial Contraction: *(*$\operatorname{codim} A_\phi = 1$*) Then ϕ is a proper modification. Again Z is $\mathbb{Q}$-factorial. If X contains only canonical (respectively terminal) singularities the singularities of Z are canonical (respectively terminal) as well.*

Small Contraction: *(*$\operatorname{codim} A_\phi \geq 2$*) Again ϕ is a proper modification. If* $\dim(X) = 3$*, Z is not $\mathbb{Q}$-*GORENSTEIN.

4.3. Relative Extremal Contractions.

THEOREM 2.36. *Let $\phi : X \to Y$ be a morphism between projective varieties with X having at worst terminal singularities. Let $H \in \operatorname{Pic}(Y)$ be ample and suppose that there exists an extremal curve $C \subset X$ with $\phi^*(H).C = 0$. If $\psi : X \to Z$ is the contraction of C, then ϕ factors through ψ, i.e. there exists a morphism $Z \to Y$ and a commutative diagram*

$$X \xrightarrow{\psi} Z \longrightarrow Y, \qquad \phi : X \to Y$$

TERMINOLOGY 2.37. The extremal contraction in theorem 2.36 is called an "relative extremal contraction".

PROPOSITION 2.38. *If $\phi : X \to Y$ is as above and there exists a curve C which is completely contained in a fiber and satisfies $K_X.C < 0$, then there exists a relative extremal contraction.*

PROOF. We use the notation as in theorem 2.36. The cone $(\phi^*(H))_{=0} \cap \overline{NE(X)}$ is an extremal subcone of $\overline{NE(X)}$. If it contains a curve C with $K_X.C < 0$, it also contains a ray which is extremal in $(\phi^*(H))_{=0} \cap \overline{NE(X)}$ and contained in $(K_X)_{<0}$ (away from 0, of course). Since $(\phi^*(H))_{=0} \cap \overline{NE(X)}$ is extremal, the ray is extremal in $\overline{NE(X)}$ as well. Now theorem 2.36 applies. □

5. Flips

DEFINITION 2.39. Let $\phi : X \to Z$ be a small MORI contraction. A commutative diagram

$$\begin{array}{ccc} X & \overset{tr(\phi)}{\dashrightarrow} & X^+ \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi^+} & \\ & Z & \end{array}$$

is called "flip of the extremal contraction ϕ" if the following holds

1. $tr(\phi)$ is birational.
2. X^+ is a normal, projective variety and $\mathbb{Q}$-GORENSTEIN.
3. ϕ^+ and $\mathrm{tr}(\phi)$ are isomorphic in codimension 1. In particular, $\mathrm{tr}(\phi) = {\phi^+}^{-1} \circ \phi$
4. K_{X^+} is ϕ^+-ample.

REMARK 2.40. Note that the signs of K_X and K_{X^+} on the exceptional sets A_ϕ respectively A_{ϕ^+} differ. Strictly speaking, while for all curves $C \subseteq A_\phi$ with $\phi(C) = (*)$ the inequality $K_X.C < 0$ holds, K_{X^+} satisfies the inverse relation $K_{X^+}.C^+ > 0$ for all curves $C^+ \subseteq A_{\phi^+}$ with $\phi^+(C^+) = (*)$.

The following theorem is due to MORI and can be found in [**Mor88**].

THEOREM 2.41. *Let X be a projective 3-dimensional variety having terminal singularities. Let $\phi : X \to Z$ be a small* MORI *contraction. Then there exists a flip.*

PROPOSITION 2.42. *Let X be as in definition 2.39. Then the following holds.*

1. *X^+ is $\mathbb{Q}$-factorial.*
2. *There exists an ample line bundle H on Z and a number m, such that the ring*
$$R := \bigoplus_{n \geq 0} (\phi_* \mathcal{O}_X(n(mK_X)) + H)$$
is finitely generated and $X^+ = Proj(R)$.
3. *If* $\dim X = 3$*, then* $\mathrm{diff}(X^+) < \mathrm{diff}(X)$*. In particular, remark that there is no infinite sequence of flips.*

TERMINOLOGY 2.43. The last proposition yields that, given a projective 3-dimensional variety X with K_X not nef, then we can find a sequence of extremal contraction and flips until we reach a variety X' admitting a contraction lowering the dimension or until $K_{X'}$ is nef. We call this procedure the "minimal model program". If $K_{X'}$ if nef, then X' is called a "minimal model". In this paper we will use the term "minimal" also for those varieties that admit a MORI contraction of fiber type.

Note that if $\phi : X \to Y$ is a relative small contraction over a variety Z, then the flipped variety X^+ still has a mapping to Z. We call this situation a "relative flip" over Z. Therefore, we can do what is called a "relative minimal model program", namely we find a sequence of relative extremal contractions and relative flips until we reach a variety X' admitting a relative contraction lowering the dimension or until $K_{X'}$ intersects all curve contained in fibers of the map $X' \to Z$ non-negatively.

LEMMA 2.44. *Let X be a projective variety with at most terminal singularities. Suppose that there is a smooth projective variety Y and a dominant birational morphism $\phi : X \to Y$. Then* $\text{diff}(X) = 0$.

PROOF. Let $\rho : \tilde{X} \to X$ be a resolution of the singularities. Let E_i^ρ, $E_i^{\phi\circ\rho} \in Div(\tilde{X})$ and $D_i^\phi \in Div(X)$ denote the ρ, $\phi \circ \rho$ and ϕ-exceptional divisors, respectively. By the definition of terminal singularities, we have the following equations of $\mathbb{Q}$-divisors:

$$K_{\tilde{X}} = \rho^*(K_X) + \sum_i a_i E_i^\rho \qquad \text{where } a_i \in \mathbb{Q}^{>0} \tag{2.1}$$

$$= \rho^*\phi^*(K_Y) + \sum_i b_i E_i^{\phi\circ\rho} \qquad \text{where } b_i \in \mathbb{N}^{>0} \tag{2.2}$$

$$K_X = \phi^*(K_Y) + \sum_i c_i D_i^\phi \qquad \text{where } c_i \in \mathbb{Q} \tag{2.3}$$

$$K_{\tilde{X}} = \rho^*\phi^*(K_Y) + \sum_i c_i\rho^*(D_i^\phi) + \sum_i a_i E_i^\rho \qquad 2.3 \text{ in } 2.1 \tag{2.4}$$

$$\sum_i b_i E_i^{\phi\circ\rho} = \sum_i c_i\rho^*(D_i^\phi) + \sum_i a_i E_i^\rho \qquad 2.4 = 2.2 \tag{2.5}$$

Looking more closely at the last equation, we can split the sum on the left hand side in those summands that are ρ-strict transforms of the D_i^ϕ and those that are ρ-exceptional, as on the right side. Comparison of coefficients yields that $a_i \in \mathbb{N}^{>0}$. So $\text{diff}(X) = 0$ by definition. □

COROLLARY 2.45. *Let X and Y be as in lemma 2.44. Assume furthermore that* $\dim X = 3$. *Then there is no small* MORI *contraction of X.*

CHAPTER 3

Mori Theory and Group Actions

1. Equivariance

The aim of this chapter is to establish equivariance results for all the steps of the minimal model program and for resolutions of singularities and equivariant rational maps. Very general results concerning resolutions can be found in [**BM96**]. The proof in the almost homogeneous context is not very difficult so that we give it here. At no point of this section we need that G is a *linear* group; we assume G to be algebraic only.

1.1. Blowing Up.

PROPOSITION 3.1. *Let X be a G-variety and $\mathcal{J}$ be a coherent sheaf of ideals on X. Assume that $\mathcal{J}$ is stable under the action of G. Then G acts on the blow-up $\tilde{X}$ of $\mathcal{J}$ and the canonical map $\pi : \tilde{X} \to X$ is equivariant.*

PROOF. This is a reformulation of [**Har93**, corollary 7.15 on p. 165]. □

COROLLARY 3.2. *Let $C \subset X$ be a G-stable subvariety. Then G acts on the blow-up $\tilde{X}$ of X with center C and the canonical map is equivariant.*

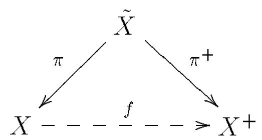
Although the existence of equivariant resolutions is known in general, the case of terminal singularities is particularly easy.

COROLLARY 3.3. *Let X be a projective variety, $\dim X = 3$, with at worst terminal singularities. Then there exists an equivariant resolution of singularities.*

PROOF. By HIRONAKA, we can desingularize X by repeatedly blowing up smooth subvarieties contained in the singular locus. Since X has at most terminal singularities, the singular set is discrete. So all singular points are G-fixed. Remember (cf. remark 2.20 on page 14) that terminal singularities are preserved under blowing up. □

DEFINITION 3.4. A rational map between G-spaces is called equivariant if it is equivariant wherever it is defined.

PROPOSITION 3.5. *Let $f : X \dashrightarrow^{eq} X^+$ be a birational mapping of 3-dimensional varieties with terminal singularities. Assume that f is equivariant with respect to the algebraic action of a connected group G. Then there exists an equivariant blow-up $\pi : \tilde{X} \to X$ dominating X^+ over f, i.e. a commutative diagram*

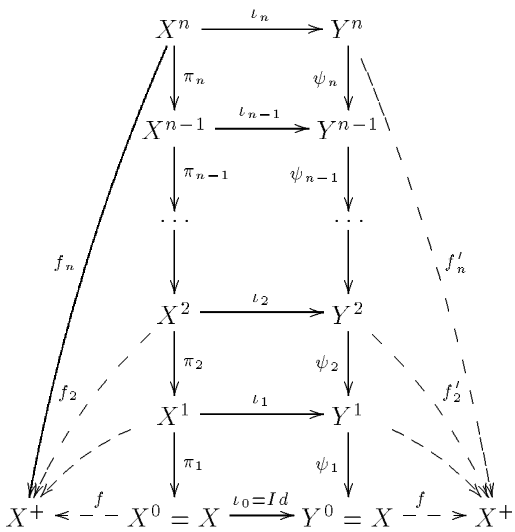


PROOF. By corollary 3.3, X can always be desingularized by a sequence of equivariantly blowing up. We may therefore assume that X is already smooth. By HIRONAKA there exists a sequence of blow-ups

$$X^n \xrightarrow{\pi_n} X^{n-1} \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_2} X^1 \xrightarrow{\pi_1} X^0 = X,$$

where the mappings π_i are blow-ups of smooth subvarieties and $\pi_n \circ \dots \pi_1 \circ f$ is regular. We must show that this can be done in an equivariant fashion. For this purpose we inductively define varieties Y_k and mappings ψ_k, ι_k such that

1. the following diagram commutes:



2. ψ_k are either equivariant blow-ups with smooth center or the identity.

We will finally show that $X^n = Y^n$ and that ι_n is the identity.

Start of Induction: Choose $Y^0 = X$, ι_0 the identity.
Induction Step: Suppose

$$Y^k \xrightarrow{\psi_k} Y^{k-1} \longrightarrow \dots \xrightarrow{\psi_1} Y^0$$

and the $\iota_0, \dots, \iota_k$ have already been constructed. π_{k+1} is the blow-up of a subvariety $C_k \subset X^k$. We consider the following cases:

$\iota_k(C_k)$ is not G-stable: In this case we set $Y^{k+1} := Y^k$ and $\psi^{k+1} := Id$. Let us remark that C_k is contained in the set of fundamental points of f_k. This set, however, is of codimension at least 2. If C_k were a curve, it was automatically G-stable. Now, if $\iota_k(C_k)$ is a curve, C_k is a curve. By equivariance of ι_k, and our last remark, we obtain that in our special case $\iota(C_k)$ is a point.

$\iota_k(C_k)$ is G-stable: In this case we show that $\iota_k(C_k)$ is smooth. Assuming this for a moment, let $\psi^{k+1} : Y^{k+1} \to Y^k$ be the blow-up of $\iota_k(C_k)$. The universal property of the blow-up (see e.g. [**Har93**, p. 164]) guarantees the existence of the ι_{k+1}.

Assume for a moment that $\iota_k(C_k)$ was singular with singular set S. By construction, this is possible only if there exist numbers $k_i \in \mathbb{N}$ with $x_{k_i} = \iota_{k_i}(C_{k_i})$ being non G-stable points. In particular, there exists $i \in \mathbb{N}$ such that

$$\psi_k^{-1} \circ \ldots \circ \psi_{k_i}^{-1}(x_{k_i}) \cap S \neq \emptyset,$$

because ι_k is isomorphic outside of

$$\bigcup_i \psi_k^{-1} \circ \ldots \circ \psi_{k_i}^{-1}(x_{k_i})$$

The set S, however, is finite and G-stable. Hence it is fixed by G, and because of equivariance, so are the ψ-images. In particular, x_{k_i} is fixed, so it was blown up in the first place. This is a contradiction to the choice of i. So S is empty.

Define $f'_k := f \circ \psi_1 \circ \ldots \circ \psi_k : Y^k \dashrightarrow X^+$. Our aim is to show that f'_n is a morphism, i.e. the set $T(f'_n)$ of fundamental points is empty. We assume to the contrary! Then we always have numbers $k_1 < \ldots < k_l$ such that ψ_{k_i+1} is the identity, which means that $\iota_{k_i}(C_{k_i})$ was not G-stable. As observed above, $\iota_{k_i}(C_{k_i})$ cannot be a curve. So it is a point x_{k_i}.

Since, however, X^n, f and Y^n, f'_n agree, if restricted to

$$X^n \setminus \bigcup_i \left(\pi_n^{-1} \circ \ldots \circ \pi_{k_i+1}^{-1}(C_{k_i})\right)$$

and

$$Y^n \setminus \bigcup_i \left(\psi_n^{-1} \circ \ldots \circ \psi_{k_i+1}^{-1} \iota_{k_i}(C_{k_i})\right).$$

respectively, we have that $T(f'_n) \subset \bigcup_i \psi_n^{-1} \circ \ldots \circ \psi_{k_i+1}^{-1} \iota_{k_i}(C_{k_i})$. In other words, there exists an i with the property that x_{k_i} is an isolated point of $\psi_{k_i+1} \circ \ldots \circ \psi_n T(f'_n)$. Note that in particular x_{k_i} is G-stable. This is a contradiction to the construction of the Y^{k_i+1}. □

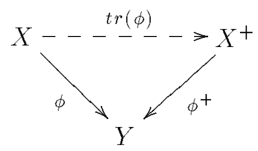
1.2. Mapping down.

Proposition 3.6. *Let X be projective with terminal singularities and $\phi : X \to Y$ be a* Mori *contraction. Then G acts on Y and ϕ is equivariant.*

Proof. All morphisms with connected fibers between normal varieties are equivariant. See [**HO80**, p. 14 and p. 16] for a proof. □

1.3. Flipping.

Proposition 3.7. *Let X be a projective 3-dimensional variety having at most terminal singularities. Let*



be a flip of X. Assume that there is an algebraic action of a connected group G on X. Then there is an algebraic G-action on X^+, and $tr(\phi)$ is equivariant wherever it is defined.

PROOF. A flip is constructed as

$$X^{+} = Proj\left(\bigoplus_{n>0} K_Y^{\otimes n}\right).$$

See [**Sho86**, p. 601] for more information. If Y_{reg} denotes the set of regular points in Y, then for all $g \in G$, $\wedge^3 Tg : K_{Y_{\mathrm{reg}}} \to K_{Y_{\mathrm{reg}}}$ is a canonical G-linearization of $K_{Y_{\mathrm{reg}}}$, inducing a degree-preserving G-linearization of the reflexive sheaf K_Y and hence of $\oplus_{n>0} K_Y^{\otimes n}$. So there is a G-action on X^+. Since the canonical mapping $X^+ \to Y$ is isomorphic at points where K_Y is a line bundle, we have the claimed equivariance. □

2. Existence of Extremal Contractions

PROPOSITION 3.8. *Let X be a projective variety, almost homogeneous with respect to an algebraic group action and which has at most terminal singularities. Then there exists a* MORI *contraction.*

PROOF. Let $\pi : \tilde{X} \to X$ be an equivariant resolution of the singularities of X. By almost homogeneity, we can always find $\dim X$ elements $v_1, \dots, v_n$ of the LIE-algebra $Lie(G)$ such that the associated vector fields

$$\tilde{v_i}(x) = \left.\frac{d}{dt}\right|_{t=0} \exp(tv_i)x \quad \in H^0(\tilde{X}, T\tilde{X})$$

are linearly independent at generic points of $\tilde{X}$. In other words,

$$\sigma := \tilde{v_1} \wedge \ldots \wedge \tilde{v_n}$$

is a non-trivial holomorphic section of the anticanonical bundle $-K_{\tilde{X}}$. In effect, we have shown that $-K_{\tilde{X}}$ is effective. Let r be the index of X. By definition of terminal singularities ($rK_{\tilde{X}} = \pi^*(rK_X) + \sum_i a_i E_i$, where $a_i \in \mathbb{N}^+$), the line bundle rK_X is effective. We still have to exclude the case that rK_X is trivial. Again by definition of terminal singularities, this can happen only if $X = \tilde{X}$, i.e. X was smooth. Assume that this is the case. The section σ nowhere vanishing implies that X is homogeneous, which in turn together with $b_1(X) = 0$ and X being projective implies that X is rational ([**HO81**]). Now either $X \cong \mathbb{P}_3$, and thus FANO, or there has to be a G-stable set of fundamental points of $X \dashrightarrow \mathbb{P}_3$, contradicting X being homogeneous.

In consequence rK_X is effective and not trivial. So there is always a curve C intersecting an effective divisor of $|-K_X|$ transversally. Hence $C.K_X < 0$. □

COROLLARY 3.9. *If X is an almost homogeneous 3-dimensional projective variety with at most terminal singularities and $\phi : X \to \mathbb{P}_1$ an equivariant morphism. Then the prerequisites of proposition 2.38 on page 18 are automatically fulfilled, i.e. there is a relative contraction over Y.*

PROOF. If $\eta \in Y$ generic, we know that the fiber X_η is smooth, does not intersect the singular set and is almost homogeneous with respect to the isotropy group G_η. So there exists a curve $C \subset X_\eta$ with $C.K_{X_\eta} < 0$. By adjunction formula lemma 2.24, $K_{X_\eta} = K_X|_{X_\eta}$. □

We have seen that all the steps of the MORI minimal model program (i.e. extremal contractions and flips) can be performed in an equivariant way. Since we can always MORI-contract an almost homogeneous variety having at most terminal singularities, we will eventually arrive at a contraction having an image of dimension < 3.

We will investigate these minimal models in the part 2 of the thesis. For this, we first fix some notation.

Notation 3.10.

- $\phi : X \to Y$ will always denote the dimension-reducing last contraction of the minimal model program.
- If $\pi : X \to Z$ is an equivariant map, then we denote $\pi(G)$ the image of the group as it is acting on Z. If $\eta \subset X$ or $\eta \subset Y$, we denote by G_η the stabilizer of η or π^{-1} of the stabilizer of η, respectively.
- If $\pi : X \to Z$ is any map and $\eta \subset Z$, we set $X_\eta := \pi^{-1}(\eta)$.

Part 2

The Minimal Models

CHAPTER 4

The case that Y is a curve

We use the notation given in 3.10 on page 25. In the case that $\dim(Y) = 1$, since Y is normal (see theorem 2.35 on page 17), it is smooth. So $Y \cong \mathbb{P}_1$ by corollary 2.7 on page 12. If $\eta \in Y$ is in the open orbit, we know —since the singularities of X are isolated— the fiber X_η is smooth. Obviously, X_η is almost homogeneous with respect to the action of the isotropy group G_η. By the adjunction formula, $K_{X_\eta} = K_X|_{X_\eta}$. All curves on X_η lie in the same extremal ray: their homology classes are identical up to multiplication by positive rational numbers, which implies that all curves in X_η intersect K_{X_η} negatively. Since X_η is almost homogeneous, it follows that it is a DEL PEZZO surface.

A simple corollary to the homological equivalence of curves in X_η is the following:

COROLLARY 4.1. *Let D be an irreducible divisor on X. Take $\eta \in Y$. If $D \cap X_\eta$ is a nonempty divisor in X_η, it intersects positively every curve which is contained in X_η.*

PROOF. There is a curve $C \subset X_\eta$ intersecting D properly. So $C.D > 0$. Let C' be any other irreducible curve in X_η. Since there exist positive numbers a and b such that the homology classes satisfy $a[C] = b[C']$, it follows that $D.C' = \frac{a}{b} D.C > 0$ □

TERMINOLOGY 4.2. We will refer to corollary 4.1 as the "homology argument".

We will show that X is isomorphic to either a linear $\mathbb{P}_2$ bundle over $\mathbb{P}_1$ or a unique minimal quadric bundle. In the latter case, we describe X explicitly, i.e. we can give the defining equations.

The general strategy to exclude the other possibilities is to use the homology argument: we construct a divisor D on X which intersects X_η but does not intersect a special curve on X_η, thus deriving a contradiction. Since X_η is a DEL PEZZO surface, it could a priori be isomorphic to either $\mathbb{P}_2$, $\mathbb{P}_1 \times \mathbb{P}_1$ or a blow-up of the latter. We will treat the cases separately.

1. X_η is isomorphic to a blown up $\mathbb{P}_1 \times \mathbb{P}_1$

This case is easiest to handle. We just need the following lemmas.

LEMMA 4.3. *Let C be a (possibly reducible) curve in X_η which is stable under the isotropy group G_η. Set $D := \overline{G.C}$. Then* $\dim D = 2$.

PROOF. By the theorem of CHEVALLEY [**Hum75**, p. 23], $G.C$ is constructible, so it contains a subset which is open in its closure. We must show that $\dim(G.C) = 2$. Clearly, $\dim(G.C) > 1$, because the orbit $G.\eta \subset Y$ is 1-dimensional. Suppose that $\dim(G.C) = 3$. If this is the case, take a general $\nu \in Y$. Then we have that

$G.C \cap X_\nu$ is 2-dimensional and thus dense in X_ν. Let g be any element of G mapping ν to η. For dimensionality reasons there is a $y \in G.C \cap X_\nu$ that is not mapped into C by g. On the other hand, by choice of y there exist a $g' \in G$ mapping η to ν and an x in C such that $g'.x = y$. So $g^{-1} \circ g'$ is an automorphism of X_η that does not stabilize C. This is a contradiction to the choice of C. □

LEMMA 4.4. *Let D be defined as above. The intersection of D with X_η is the curve C:*

$$D \cap X_\eta = C.$$

PROOF. Suppose this is not the case. Then there exists a curve $\tilde{C} \subset (D \setminus C) \cap X_\eta$. The analogous argument to the above shows that $G.\tilde{C}$ is 2-dimensional (as a constructible set). Because D is G-stable we know that $G.\tilde{C} \subseteq D \setminus G.C$. However, we know already that $\dim(G.C) = 2$, D irreducible, and that $G.C$ is open in its ZARISKI closure, contrary to the irreducibility of D. □

COROLLARY 4.5. *If η is in the open orbit of Y, then X_η is either $\mathbb{P}_2$ or $\mathbb{P}_1 \times \mathbb{P}_1$.*

PROOF. Suppose to the contrary that X_η is a blown up $\mathbb{P}_1 \times \mathbb{P}_1$. Let C_i denote the blown up curves in X_η, $C := \cup_i C_i$ and D as above. Then there are curves in X_η that do not intersect D. So we have derived a contradiction to corollary 4.1. □

2. X_η is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$

This case requires substantially more work. We start with a brief investigation of the action of one parameter groups on X.

Identify X_η with $\mathbb{P}_1 \times \mathbb{P}_1$, and let π_1 and $\pi_2 : X_\eta \to \mathbb{P}_1$. be the standard projections. A curve which is mapped to a point by π_1 will be said to be "vertical with respect to a given identification $X_\eta \cong \mathbb{P}_1 \times \mathbb{P}_1$. A "horizontal curve" is analogously defined as a π_2-fiber.

LEMMA 4.6. *Let $H < G$ be a 1-dimensional algebraic subgroup of G acting non-trivially on Y. Then there exists an element $h \in H_\eta$ mapping horizontal curves to vertical curves.*

PROOF. Let C be a horizontal curve. Define the divisor $D := \overline{H.C}$ as above. H_η is zero-dimensional and finite —since H is algebraic. Recall that, since η is generic, $D \cap X_\eta = H.C \cap X_\eta = H_\eta C$. The curve $C' := H_\eta.C$ is the union of finitely many curves. If $D_\eta = D \cap X_\eta$ is a union of finitely many horizontal curves, then D_η does not intersect a general horizontal curve, contrary to corollary 4.1. Thus there exists $h \in H_\eta : h.C$ is not horizontal. If $h.C$ was non-vertical, take a disjoint horizontal curve C'. The curves $h.C$ and $h.C'$ are disjoint. This, however, is not possible if one of them was neither horizontal nor vertical. □

LEMMA 4.7. *Let H be an algebraic 1-dimensional subgroup of G acting non-trivially on Y. Then $H \ncong \mathbb{C}$.*

PROOF. By lemma 4.6 there is an $x \in X$ such that $H.x$ is an unramified connected cover over $H.\phi(x)$ with more than one leaf. However, a $\mathbb{C}$ orbit on $Y \cong \mathbb{P}_1$ is isomorphic to $\mathbb{C}$, i.e. it is simply connected. □

REMARK 4.8. Let $F := \mathrm{Ker}(\phi : G \to \mathrm{Aut}(Y))$. The above shows that there is no 1-parameter group in $\mathrm{Im}(\phi : G \to \mathrm{Aut}(Y))$ which is isomorphic to $\mathbb{C}$. Sine the maximal torus of $\mathrm{Aut}(\mathbb{P}_1)$ is 1-dimensional, it follows that $\mathrm{Im}(\phi : G \to \mathrm{Aut}(Y)) \cong$

$\mathbb{C}^*$. Therefore G fixes exactly 2 points in Y. All the elements in the isotropy group G_η fix an additional point. There is, however, only one automorphism in $\mathrm{Aut}(Y)$ fixing more than 2 points: the identity. Therefore $F = G_\eta$. Hence we know that F acts almost transitively on the generic fiber. The same holds for F^0.

Lemma 4.9. *Take a generic fiber X_η and identify it with $\mathbb{P}_1 \times \mathbb{P}_1$. Then F does not contain a subgroup which is isomorphic to $\mathbb{C}^*$ and which only acts in horizontal (resp. vertical) direction.*

Proof. Suppose to the contrary and let T^* be the group. We claim that the set $D := \mathrm{Fix}(T^*)$ of fixed points is a divisor. Now $D \cap X_\eta$ are two curves. We assume without loss of generality that they are horizontal. We linearize the T^* action at one of these points: after suitable choice of coordinates, T^* acts by

$$T^* : \lambda(x, y, z) \to (\lambda^{n_1} x, \lambda^{n_2} y, z)$$

and $\{x = y = 0\}$ coincides with the T^* fixed curve. We know that T^* does not act on the base Y, so that without loss of generality $n_2 = 0$. This immediately implies that locally $D = \{x = 0\}$, so D is indeed a divisor.

There are other horizontal curves in X_η which do not intersect D. This is a contradiction to the homology argument! □

Lemma 4.10. *The stabilizer F^0 of X_η is not isomorphic to $(\mathbb{C}^2, +)$.*

Proof. We assume to the contrary. Let $H^* < G$ be a group which is isomorphic to $\mathbb{C}^*$, acting non-trivially on Y. Then there is a group $B < F$, chosen to be normalized by H^* such that $B \cong \mathbb{C}$ and $B < \mathrm{Ker}(\pi_1 : F \to \mathrm{Aut}(\mathbb{P}_1))$. By lemma 4.6 on the preceding page, H^*_η is finite, not trivial and acts non-trivially on the fibers.

The group F is normal in G. So H^*_η fixes the unique F-fixed point. Hence we know that H^*_η stabilizes the union of the horizontal and vertical curves through that point. So H^*_η stabilizes the complement which is the open orbit of F in X_η. This orbit is isomorphic to $\mathbb{C} \times \mathbb{C}$. We can identify F with $\mathbb{C} \times \mathbb{C}$ in a way that B becomes $(0, \mathbb{C})$. H^* acts on F by conjugation. We write

$$z^{-1}(0, b)z = (f(z), g(z))$$

where f and g are continuous functions with $f(1) = 0$ and $g(1) = b$.

Choose an arbitrary point x_0 in the open orbit of X_η. Let $\{x_i\} = H^*_\eta.x_0$ and $C := \overline{H^*.x_0}$. Note that all the $x_i \in X_\eta$. Furthermore, we define the divisor D to be $D := \overline{B.C}$. Again by lemma 4.6, D cannot intersect X_η in horizontal curves only. So there exists a generic point $y_0 \in D \cap X_\eta \setminus \overline{\bigcup_i B.x_i}$. Let $\{y_i\} = H^*_\eta.y_0$.

Choose $U \subset X_\eta$ to be the union of 2-dimensional polycylinders around the y_i. We may take U small so that $\overline{U} \cap \overline{\bigcup_i B.x_i} = \emptyset$. We set

$$V' := \{z \in H^* | \forall b \in B : \forall i : z^{-1}bz(x_i) \cap \overline{U} = \emptyset\}.$$

The continuity of f and g guarantees that V' is open and not empty. We can take an open subset $V \subset V'$ in order to have $V \cap H^*_\eta = \{1\}$ and $\pi_1(V) = 1$.

The map

$$\begin{array}{rcl} m : U \times V & \to & X \\ (u, v) & \to & v.u \end{array}$$

has a Jacobian of maximal degree. Thus m is locally biholomorphic, implying that m is locally open. After shrinking V and U once more, we may therefore assume that $m(U \times V)$ is open.

By CHEVALLEY's theorem we know that $B.H^*$ is a group which has an open and dense orbit at x_0 in D. For that reason we can always find a $w \in (B.H^* x_0) \cap m(U \times V)$.

In other words, there exist $b \in B$ and $z' \in H^*$ such that $b.z'.x_0 = \omega$. By replacing x_0 with one of the x_i, we can take $z \in V$ instead of $z' \in H^*$. Recall that V was chosen to intersect H^*_η only once. Hence z, chosen as above, is unique and $z^{-1}\omega \in U$. So $z^{-1}bzx_i \in U$, which is a contradiction to the choice of V!

□

PROPOSITION 4.11. *The connected subgroup F^0 is isomorphic either to SL_2 or to a* BOREL *subgroup $B < SL_2$ and acts almost transitively, i.e. diagonally on X_η.*

PROOF. We claim that if F^0 does not have a fixed point on X_η, then it is isomorphic to SL_2 and acts diagonally. We consider a LEVI-MALCEV-decomposition[1] $F^0 = S \ltimes R$. If S was trivial, i.e. F^0 solvable, F^0 had a fixed point by BOREL's fixed point theorem. So S is non-trivial.

If some SL_2 in S acts only in one factor, then the standard *horizontal-vertical* argument using the closure of an $H^*.Sl_2$-orbit leads to a contradiction. Thus $S = Sl_2$ and its action is diagonal. It has two-orbits, the diagonal and its complement, and each must be stabilized by the radical R of F^0, because otherwise F^0 would act transitively and would be semi-simple. But the R-action on the diagonal is trivial, because S acts transitively, and thus R acts trivial on the fiber X_η. Consequently, if F^0 is not solvable, then it is semi-simple.

The other case is that F^0 has a number of fixed points. The possible configurations are shown in figure 1. Here circles mean isolated fixed points and solid lines represent lines of fixed points. We will exclude most of the the different configurations case by case, using lemma 4.9.

- **The cases (δ) and (ϵ) do not occur:** F^0 is normal in G. For that reason, all the $g \in G$ map F^0 orbits to F^0 orbits, and $G_\eta < G$ stabilizes the set of fixed points. A contradiction to lemma 4.6 on page 30.
- **The case (γ) does not occur:** We have to have $F^0 \cong \mathbb{C}^* \times \mathbb{C}^*$. Consequently there exists a subgroup $I < F$, $I \cong \mathbb{C}^*$ such that for $i \in \{1, 2\}$: $I^* < \mathrm{Ker}(\pi_i)$, and we have a contradiction as in lemma 4.9.
- **The case (β) does not occur:** Let C be a horizontal curve through the fixed points. G_η stabilizes the curve C. We obtain a contradiction to lemma 4.6 on page 30.

The case (α) is indeed possible as we will see in example 4.12. Here F^0 can act almost transitively only if it is isomorphic to a BOREL subgroup of SL_2 and acts diagonally. □

EXAMPLE 4.12. Consider the space $V_0 := \mathbb{P}_3 \times \mathbb{C}$ with coordinates $([x_0 : x_1 : x_2 : x_3], z)$. Define the subvariety $X^0 := \{x_0^2 + x_1^2 + x_2^2 = x_3^2 z\}$. Let the group SL_2 act on the first component only: it's action is the standard action of $SO_3 \cong SL_2/\{\pm I\}$ on $\mathbb{C}^3$, extended to $\mathbb{P}_3$.

[1]We abuse notation here because the radical and the semisimple part may have finite intersection.

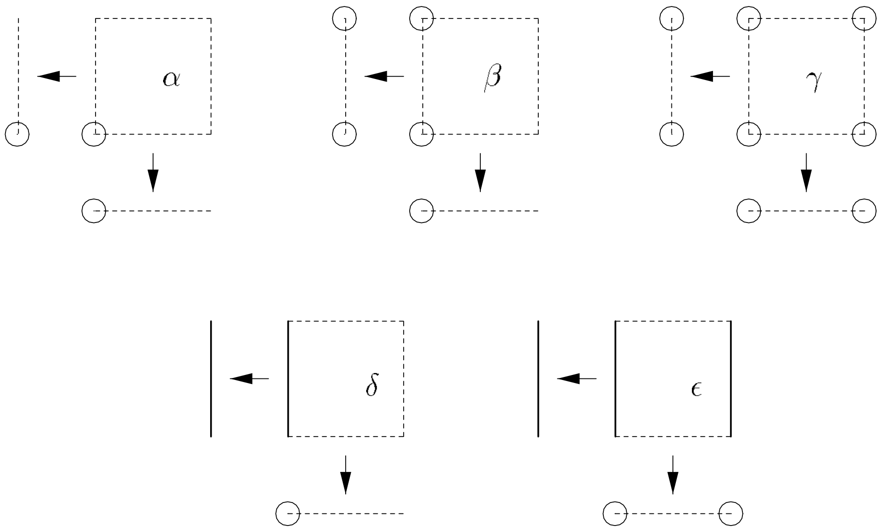


FIGURE 1. Possible fixed point configurations

Let the group $\mathbb{C}^*$ act as follows:

$$\lambda([x_0 : x_1 : x_2 : x_3], z) = ([\lambda x_0 : \lambda x_1 : \lambda x_2 : x_3], \lambda^2 z).$$

So, as a simple calculation shows $G := \mathbb{C}^* \times SL_2$ acts and stabilizes X.

We can construct a similar quasi-projective variety X_∞ over $\mathbb{C}$: Again $V_\infty := \mathbb{P}_3 \times \mathbb{C}$ and $X^\infty := \{x_0^2 + x_1^2 + x_2^2 = x_3^2 z\}$. Let SL_2 act as above and $\mathbb{C}^*$ by:

$$\lambda([x_0 : x_1 : x_2 : x_3], z) = ([x_0 : x_1 : x_2 : \lambda x_3], \lambda^{-2} z).$$

The last step of our construction consists in gluing both V_0 and V_∞ together in order to obtain a $\mathbb{P}_3$-bundle over $\mathbb{P}_1$ containing the desired almost homogeneous space which is the corresponding gluing together of X^0 and X^∞. Define the equivalence relation

$$\begin{aligned} &V_0 \ni ([x_{0,0} : x_{0,1} : x_{0,2} : x_{0,3}], z_0) \sim ([x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3}], z_\infty) \in V_\infty \\ &:\Leftrightarrow z_0 z_\infty = 1 \text{ and } [x_{0,0} : x_{0,1} : x_{0,2} : x_{0,3}] = [x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3} z_\infty]. \end{aligned}$$

There are still several things to show:

X^0 and X^∞ glue together to a variety X: We take the equation defining X^0 and substitute the equivalent coordinates of V_∞:

$$\begin{aligned} x_{0,0}^2 + x_{0,1}^2 + x_{0,2}^2 &= x_{0,3}^2 z_0 \\ \Leftrightarrow x_{\infty,0}^2 + x_{\infty,1}^2 + x_{\infty,2}^2 &= (x_{\infty,3} z_\infty)^2 \frac{1}{z_\infty} \\ \Leftrightarrow x_{\infty,0}^2 + x_{\infty,1}^2 + x_{\infty,2}^2 &= x_{\infty,3}^2 z_\infty \end{aligned}$$

which is the equation defining V_∞.

X is smooth: Since X has the same equation in V_0 and V_∞, it is sufficient to prove smoothness of X^0. We have 4 coordinate patches, namely $\{x_i \neq 0\}$. We check

$\{x_3 \neq 0\}$: Here $V_0 = \{P = 0\}$ with $P = x_0^2 + x_1^2 + x_2^2 - z$. We have

$$\begin{pmatrix} \frac{\partial P}{\partial x_0} \\ \frac{\partial P}{\partial x_1} \\ \frac{\partial P}{\partial x_2} \\ \frac{\partial P}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x_0 \\ 2x_1 \\ 2x_2 \\ -1 \end{pmatrix} \neq 0 \text{ on } V_0 \cap \{x_3 \neq 0\}$$

$\{x_2 \neq 0\}$: Here $V_0 = \{P = 0\}$ with $P = x_0^2 + x_1^2 + 1 - zx_3^2$. We have

$$\begin{pmatrix} \frac{\partial P}{\partial x_0} \\ \frac{\partial P}{\partial x_1} \\ \frac{\partial P}{\partial x_3} \\ \frac{\partial P}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x_0 \\ 2x_1 \\ -z2x_3 \\ -x_3^2 \end{pmatrix} \neq 0 \text{ on } V_0 \cap \{x_3 \neq 0\}$$

$\{x_0 \neq 0\}$ and $\{x_1 \neq 0\}$: These cases are handled similarly.

X is G-almost homogeneous: It is obvious that the SL_2-action is the same on V_0 and V_∞, i.e. if $g \in SL_2$ and $v_0 \sim v_1$ then $g.v_0 \sim g.v_1$. If $([x_{0,0} : x_{0,1} : x_{0,2} : x_{0,3}], z_0) \sim ([x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3}], z_\infty)$ and $\lambda \in \mathbb{C}^*$, then

$$\begin{aligned} \lambda([x_{0,0} : x_{0,1} : x_{0,2} : x_{0,3}], z_0) &= ([\lambda x_{0,0} : \lambda x_{0,1} : \lambda x_{0,2} : x_{0,3}], z_0\lambda^2) \\ \lambda([x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3}], z_\infty) &= ([x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : \lambda x_{\infty,3}], z_\infty\lambda^{-2}) \end{aligned}$$

Now note that $(z_0\lambda^2)(z_\infty\lambda^{-2}) = z_0 z_\infty$ and

$$\begin{aligned} [x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : \lambda x_{\infty,3} z_\infty \lambda^{-2}] &= [\lambda x_{\infty,0} : \lambda x_{\infty,1} : \lambda x_{\infty,2} : x_{\infty,3} z_\infty] \\ &= [\lambda x_{0,0} : \lambda x_{0,1} : \lambda x_{0,2} : x_{0,3} z_\infty] \end{aligned}$$

if $[x_{0,0} : x_{0,1} : x_{0,2} : x_{0,3}] = [x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3} z_\infty]$.

X can be Mori-contracted to $\mathbb{P}_1$: We can find a relative Mori contraction $\psi : X \to Z$ over $\mathbb{P}_1$ (see e.g. corollary 3.9 on page 24). Since X is smooth, the contraction cannot be small.

Note that if X_μ is an arbitrary fiber of the map $X \to \mathbb{P}_1$, then all curves contained in X_μ are equivalent as homology cycles: this is clear for the singular fibers over 0 and ∞ because they are singular quadrics, and also true for the generic fibers because the action of $\pm 1 \in \mathbb{C}^*$ swaps horizontal and vertical directions. Therefore ψ-fibers are also fibers of the map $X \to \mathbb{P}_1$. As a last step we have to exclude that ψ is just the blowing down of a finite set of fibers. This, however cannot happen: if X_μ is mapped to a point by ψ, then nearby fibers $X_{\mu+\epsilon}$ are mapped into a Stein neighborhood of $\psi(X_\mu)$, which is possible only if $X_{\mu+\epsilon}$ is a fiber itself.

As an overall result we obtain that the all the ψ fibers coincide with the fibers of the map $X \to \mathbb{P}_1$.

Notation 4.13. We will refer to the variety described in example 4.12 as the "minimal quadric bundle".

Lemma 4.14. *If X is the minimal quadric bundle described in example 4.12, then there is a G-equivariant rational map $X \dashrightarrow^{eq} \mathbb{P}_3$.*

PROOF. Consider the map

$$\begin{array}{rcl}\pi_1 : V_0 & \to & \mathbb{P}_3 \\ ([x_{0,0} : x_{0,1} : x_{0,2} : x_{0,3}], z_0) & \to & [x_{0,0} : x_{0,1} : x_{0,2} : x_{0,3}]\end{array}$$

This mapping is clearly equivariant with respect to SL_2. It is clear that the map extends to a rational map $\pi : X \to \mathbb{P}_3$. Therefore, the only thing we have to show is the equivariance of the $\mathbb{C}^*$-action. We calculate the π-images of $([x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3}], z_\infty) \in V_0 \cap V_\infty$. We know that

$$([x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3}], z_\infty) \in V_0$$

is equivalent to

$$\left([x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3} z_\infty], \frac{1}{z_\infty}\right).$$

Hence

$$\pi([x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3}], z_\infty) = [x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3} z_\infty].$$

$\lambda \in \mathbb{C}^*$ acts on $\mathbb{P}_3$ by definition as

$$\lambda([x_{0,0} : x_{0,1} : x_{0,2} : x_{0,3}]) = [\lambda x_{0,0} : \lambda x_{0,1} : \lambda x_{0,2} : x_{0,3}]$$

hence

$$\begin{array}{rcl}\pi\lambda([x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3}], z_\infty) & = & \pi([x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : \lambda x_{\infty,3}], \lambda^{-2} z_\infty) \\ & = & [x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : (\lambda x_{\infty,3})(\lambda^{-2} z_\infty)] \\ & = & [\lambda x_{\infty,0} : \lambda x_{\infty,1} : \lambda x_{\infty,2} : x_{\infty,3} z_\infty]\end{array}$$

On the other hand

$$\begin{array}{rcl}\lambda\pi([x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3}], z_\infty) & = & \lambda[x_{\infty,0} : x_{\infty,1} : x_{\infty,2} : x_{\infty,3} z_\infty] \\ & = & [\lambda x_{\infty,0} : \lambda x_{\infty,1} : \lambda x_{\infty,2} : x_{\infty,3} z_\infty].\end{array}$$

Therefore π is equivariant and rational. □

PROPOSITION 4.15. *Let X be an almost homogeneous 3-dimensional projective variety with at worst terminal singularities and assume that there is an extremal contraction $X \to \mathbb{P}_1$ with the generic fiber being isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. Then X is equivariantly embedded in a $\mathbb{P}_3$-bundle over $Y \cong \mathbb{P}_1$.*

PROOF. X has terminal singularities. These are rational by the theorem of ELKIK and FLENNER (see [**Rei87**, p. 363]). Therefore X is COHEN-MACAULAY (see [**Rei87**, p. 369]). Since ϕ has equidimensional fibers, ϕ is flat, as shown in [**Har93**, p. 276].

Let $\mu \in Y$ be a generic point and let D_μ be the diagonal in $X_\mu \cong \mathbb{P}_1 \times \mathbb{P}_1$. D_μ is stable with respect to $\mathrm{Ker}(\phi)$ and H^*_μ. Set $D := \overline{H^*.D_\mu}$. We have seen in lemma 2.33 on page 16 that for $k >> 0$ the line bundle associated to $L := D + kX_\eta$ is ample. Note that $L|_{X_\eta} = D \cap X_\eta$ is the diagonal in $X_\eta \cong \mathbb{P}_1 \times \mathbb{P}_1$. Hence

$$(L_{X_\eta})^2 = 2. \tag{4.1}$$

We also know that $h^0(X_\mu, L_{X_\mu}) = 4$. This enables us to calculate the Δ-genus of X_η:

$$\Delta(X_\eta, L_{X_\eta}) = \dim(X_\eta) + L^2_{X_\eta} - h^0(X_\eta, L_{X_\eta}) = 0.$$

Now pick an arbitrary point $\nu \in Y$. Since ϕ is flat, (4.1) implies that $(L_{X_\nu})^2 = 2$. Since L is a flat sheaf over Y, it follows that $4 = h^0(X_\eta, L_{X_\eta}) \leq h^0(X_\nu, L_{X_\nu})$ by the semicontinuity theorem (see [**Har93**, thm. 12.8 on p. 288]).

Now $\Delta(Z, L') \geq 0$ for all varieties Z with ample line bundles L'. Thus $\Delta(X_\nu, L_{X_\nu}) = 0$ and $L^2_{X_\nu} = 2$. By FUJITA's classification results of polarized varieties (see [**BS95**, prop. 3.1.2]), we conclude that X_ν is isomorphic to a hyperquadric in some $\mathbb{P}_n$ and that $h^0(X_\nu, L_{X_\nu}) = 4$.

Since ϕ is flat and $h^0(X_\mu, L_{X_\mu})$ is constant on Y, it follows by a theorem of GRAUERT (see [**Har93**, p. 288]) that $E := \phi_*(L)$ is a vector bundle of rank 4 on Y. We will construct an embedding $\alpha : X \to \mathbb{P}(E^*)$. A basis of the sections of L_{X_μ} extends to sections $s_0, \ldots, s_3$ of L_U on a neighborhood U of X_μ, in the ZARISKI topology. Since the sections $s_0, \ldots, s_3$, restricted to X_μ span L_{X_μ}, it can be assumed by possibly shrinking U that the sections $s_0, \ldots, s_3$ have no common zeroes on U, see e.g. the discussion in [**Har93**, III, Ex. 12.7.2]. Choose a ZARISKI open set $V \subset Y$ with $X_\mu \subset \phi^{-1}(V) \subset U$. The s_k can be seen as linearly independent sections of E over V, which give a local trivialization. Using these coordinates, the map $\alpha : X \to \mathbb{P}(E^*)$, restricted to to the trivialization over V is defined by sending a point $x \in \phi^{-1}(V)$ to $(\phi(x), [s_0(x), \ldots, s_3(x)])$. This is clearly an embedding over Y.

There still remains the question of equivariance. So far we have not yet shown that G acts on $\mathbb{P}(E^*)$. A point $e \in E$ over μ is defined to be a section $\sigma_e \in H^0(X_\mu, L_{X_\mu})$. Applying $g \in G$, we obtain $g\sigma_e$ which is a section in $H^0(X_\mu, gL_{X_\mu})$. Now L and gL are isomorphic. Thus there exists a bundle isomorphism $\iota_g : gL \to L$.

A point $e^* \in E^*$ over μ is defined to be an element of $H^0(X_\mu, L_{X_\mu})^*$. One is tempted to define a G-action on E^* by setting

$$(ge^*)(e) := e^*(\iota_{g^{-1}} g^{-1} e). \tag{4.2}$$

The problem is of course that $\iota_{g^{-1}}$ is not at all unique. In fact, two bundle isomorphisms might differ by a constant non-zero factor. Therefore equation (4.2) is not well defined if $e^* \in E^*$. It is, however, well-defined on the equivalence class of e^* in $\mathbb{P}(E^*)$. Therefore G does not act on E^*; it acts on $\mathbb{P}(E^*)$ instead.

As a last step, we show that $\alpha : X \to \mathbb{P}(E^*)$ is indeed equivariant. We have to show that $g \circ \alpha = \alpha \circ g$.

$$\begin{aligned}
(g\alpha(x))(e) &= \alpha(x)(\iota_{g^{-1}} g^{-1} e) && \text{By equation (4.2)} \\
&= (\iota_{g^{-1}} g^{-1} e)(x) && \text{definition of } \alpha \\
&= (g^{-1} e)(x) && \text{constant factors don't count} \\
&= e(\iota_g g x) && \text{definition of } G\text{-action} \\
&= e(gx) && \text{constant factors don't count} \\
&= \alpha(gx)(e)
\end{aligned}$$

□

PROPOSITION 4.16. *Let X be an almost homogeneous 3-dimensional projective variety with at worst terminal singularities and assume that there is an extremal contraction $X \to \mathbb{P}_1$ with the generic fiber being isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. Then X is isomorphic to the minimal quadric bundle given in example 4.12.*

PROOF. Recall that 0 and ∞ denote the two G-fixed points in Y. By proposition 4.15 we may assume $X \subset \mathbb{P}(E^*)$ equivariantly. Let $\psi : \mathbb{P}(E^*) \to Y$

be the canonical projection. We furthermore set $U := Y \setminus \{\infty\} \cong \mathbb{C}$ and $V := \psi^{-1}(U) \cong \mathbb{P}_3 \times \mathbb{C}$. We have already shown that $X \cap (\mathbb{P}_3, 0)$ is a G-stable singular quadric with singularity S. The action of a maximal torus $T < G$ fixes S and additionally stabilizes a linear hyperplane $P \subset (\mathbb{P}_3, 0)$ with $S \notin P$.

We claim that $(\mathbb{P}_3, 0)$ does not contain a curve of B^*-fixed points. Assume for a moment that there was. Then we can find a fixed point which is smooth both in the curve and in X. Linearization of a neighborhood yields local coordinates (x_1, x_2, x_3). Without loss of generality, the local B^*-action takes the form:

$$\lambda(x_1, x_2, x_3) \to (x_1, \lambda^{n_2} x_2, \lambda^{n_3} x_3)$$

with $\{x_3 = 0\} \subset X \cap (\mathbb{P}_3, 0)$. The curve $(0, 0, \mathbb{C})$ is not contained in $(\mathbb{P}_3, 0)$. However since the B^*-orbits (and their closures) are completely contained in ϕ-fibers, we know that $(0, 0, \mathbb{C})$ is not a single B^*-orbit. In other words, $n_3 = 0$. In essence we have shown that there exists a surface of B^*-fixed points which is not contained in a single fiber. So every fiber has to have at least a curve of B^*-fixed points. This, however, is a contradiction to what is known about the B^*-action.

Since T stabilizes the intersection $Q = P \cap X_0$, there is a 1-dimensional subgroup $\tilde{T}$ of T which fixes it pointwise. Now Q is a quadratic curve in P. Thus $\tilde{T}$ fixes P pointwise. Since B has no curve of fixed points in X_0, we may take $H^* := \tilde{T}$, i.q. without loss of generality, H^* fixes P pointwise.

Proceeding with H^* chosen as above, there are four H^*-fixed points of $(\mathbb{P}_3, 0)$ lying in general position: 3 generic points in P and S will do. Linearizing the H^* action at these points, we find four H^*-stable sections σ_i which are disjoint over U. In order to show that they are even linearly independent over U (i.e. for all $\mu \in U$, the 4 points $X_\mu \cap \sigma_i$ are not contained in a linear hyperplane), we note that they are over a neighborhood of $0 \in U$. The H^*-action is of degree 1 and therefore has to preserve linear independence.

The σ_i give coordinates $([x_0 : x_1 : x_2 : x_3], z)$ on V in a way that for all i we have $\sigma_i \cap V = \{x_j = 0 | j \neq i\}$. For $\lambda \in H^*$, it follows that

$$\lambda([x_0 : x_1 : x_2 : x_3], z) = \left(\begin{pmatrix} \eta_0 & 0 & 0 & 0 \\ 0 & \eta_1 & 0 & 0 \\ 0 & 0 & \eta_2 & 0 \\ 0 & 0 & 0 & \eta_3 \end{pmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}, \lambda^n z \right),$$

where the η_i are polynomials in λ and z. Now H^* acts as a group and therefore obeys the composition law

$$\eta_i(c\lambda, z) = \eta_i(\lambda, z)\eta_i(c, \lambda^n z).$$

If the polynomial η_i contains z to the maximal power of d, then the right hand side contains z to the $2d$. Therefore $d = 0$, i.e. the η_i are independent of z. Thus $\eta_i(\lambda) = \lambda^{n_i}$.

Restricted to $(\mathbb{P}_3, 0)$, the group H^* acts as

$$\lambda[x_0, x_1, x_2, x_3] \to [\lambda^{n_0} x_0, \lambda^{n_1} x_1, \lambda^{n_2} x_2, \lambda^{n_3} x_3].$$

We may assume for simplicity that $P \neq \{x_0 = 0\}$ and that $n_0 = 0$ (note that we can always add integers to all the n_i without changing the action). If restricted to $\{x_0 \neq 0\} \cong \mathbb{C}^3$, H^* acts as

$$\lambda(x_1, x_2, x_3) \to (\lambda^{n_1} x_1, \lambda^{n_2} x_2, \lambda^{n_3} x_3).$$

The hyperplane P has non-empty intersection with $\{x_0 \neq 0\}$ and is pointwise H^*-fixed. Thus $n_1 = n_2 = 0$. Therefore $P = \{z = x_3 = 0\}$, and $S = ([0:0:0:1], 0)$ is the only remaining H^*-fixed point in $(\mathbb{P}_3, 0)$. The H^*-action at S is isotropic: all vectors in $(\mathbb{P}_3, 0) \setminus P \cong \mathbb{C}^3$ are eigenvectors with weight $-n_3$.

As a last step, we show that the equation defining X in V takes the form as claimed in the proposition. Define $D := \{x_3 = 0\}$ and note that D is H^*-stable and fixed with respect to the isotropy group H^*_η. Consider the equation defining X_1 in $(\mathbb{P}_3, 1) \setminus D \cong \mathbb{C}^3$. It takes the form $X_1 \setminus D = \{P^2 + P^1 = c\}$, where P^i are homogeneous polynomials in x_0, x_1 and x_2 of degree i, respectively. If $\xi \in H^*_\eta$ is a generating element, it stabilizes the variety $X_1 \setminus D$. In other words:

$$\begin{aligned}
\{P^2 + P^1 = c\} &= \{P^2 \circ \xi + P^1 \circ \xi = c\} \\
&= \{\xi^2 P^2 + \xi P^1 = c\} \\
&= \{\xi P^2 + P^1 = \xi^{-1} c\} \\
&\subset \{(1-\xi) P^2 = (1 - \xi^{-1}) c\}
\end{aligned}$$

The last $\subset$-sign is in fact equality, because the equation defining $X_1 \setminus D$ is quadratic, irreducible and reduced. Thus $X_1 \setminus D = \{P^2 = c\}$. If $c = 0$, then X_1 is singular at $([0:0:0:1], 1)$ which is impossible. Since $\{P^2 = c\} = \{\xi^2 P^2 = c\}$, it follows that $\xi^2 = 1$. For this reason we know that without loss of generality we may assume $n = 2$ and $n_3 \in \sqrt{1}$.

The nature of the H^* action allows us to adjust our coordinates such that X_1 takes the form $\{x_0^2 + x_1^2 + x_2^2 = 1, z = 1\}$. A generic fiber X_η is now given if we take $\lambda \in \sqrt{\eta} \subset H^*$ and use $X_\eta = \lambda X_1$. If we perform the calculations, we obtain $X_\eta = \{x_0^2 + x_1^2 + x_2^2 = \lambda^{2n_3} x_3^2, z = \eta = \lambda^2\}$ so that as a net result

$$X = \begin{cases} x_0^2 + x_1^2 + x_2^2 = z x_3^2 & \text{if } n_3 = -1 \\ z(x_0^2 + x_1^2 + x_2^2) = x_3^2 & \text{if } n_3 = 1. \end{cases}$$

In the case $n_3 = 1$. X_0 is a non-reduced $\mathbb{P}_2$, contradicting earlier results, so that $n_3 = -1$.

After a similar argumentation for the part of X over $\mathbb{P}_1 \setminus \{0\}$, we again obtain the equations of the minimal quadric bundle described in example 4.12. Since, apart from permuting the x_0, x_1 and x_2, there is no choice of how the affine parts can be glued together, X is isomorphic to our example. □

3. X_η is isomorphic to $\mathbb{P}_2$

Recall our situation: $\phi : X \to Y$ is a MORI contraction from X to $Y \cong \mathbb{P}_1$. In this section we assume that the generic fiber X_η is isomorphic to $\mathbb{P}_2$. We will use the following criterion for a variety being a bundle of projective spaces. This is taken almost verbatim from [**BS95**, Prop. 3.2.1].

THEOREM 4.17. *Let X be an n-dimensional connected projective variety and let $p : X \to Y$ be a holomorphic surjection from X onto a normal variety Y. Let L be an ample line bundle on X. Assume that $(F, L_F) \cong (\mathbb{P}_d, \mathcal{O}_{\mathbb{P}_d}(1))$ for a general fiber F of p and that all fibers of p are d-dimensional. Further assume that X is* COHEN-MACAULAY *and that Sing(X) contains no fiber of the map. Then $p : X \to Y$ gives (X, L) the structure of a linear $\mathbb{P}_d$-bundle with $X \cong \mathbb{P}(p_* L)$. In particular X is smooth if and only if Y is smooth.*

Our application is the following

THEOREM 4.18. *Let X be a projective almost homogeneous variety having at most terminal singularities. Suppose X admits a* MORI*-contraction $\phi : X \to Y$ to a curve Y with the generic fiber being isomorphic the $\mathbb{P}_2$. Then X is smooth and has the structure of a linear $\mathbb{P}_2$-bundle over $\mathbb{P}_1$.*

PROOF. We have to show the following:

All fibers are 2-dimensional: This is clear, because Y is 1-dimensional.

Existence of *L*: First we show that there is a divisor $L' \in Div(X)$ intersecting a generic fiber in a line. For this take a 1-dimensional subgroup H of G acting non-trivially on Y. We consider the following cases:

1. H_η does not act on Y. Then we can simply take a line $l \subset X_\eta$ and consider the closure of the orbit: $L' := \overline{H.L}$.
2. H_η is finite and does act on X_η. In this case we know that $H \cong \mathbb{C}^*$ and that H_η is therefore cyclic. Hence there is always a H_η-stable line L in X_η. So we set $D := \overline{H.L}$.

Take A to be an ample divisor on Y. To construct L, we use the fact that for $n >> 0$, the line bundle $L := L' + n\phi^*A$ is very ample on X (this is a consequence of the fact that $\rho(X) = \rho(Y) + 1$).

The other prerequisites of theorem 4.17 follow from the remarks in the introductory chapters. □

We close this section by showing that all the cases mentioned above really appear.

PROPOSITION 4.19. *Let X be a linear $\mathbb{P}_2$-bundle over $\mathbb{P}_1$. Then X is almost homogeneous.*

PROOF. Since X is the projective bundle $\mathbb{P}(E)$ associated to a rank 3 vector bundle which splits $E = E_1 \oplus E_2 \oplus E_3$ as a direct sum of line bundles (see [**OSS80**, p. 22]), it follows from an argument similar to [**HO80**, prop. 5 on page 18] that the SL_2-action on the base lifts to an action on E

We also have an action of $(\mathbb{C}^*)^3$ on E, and also on the quotient X. Since SL_2 acts transitively on the base and $(\mathbb{C}^*)^3$ acts almost homogeneously on the fiber, it follows that $\mathrm{Aut}(E)$ acts almost transitively on X. □

CHAPTER 5

The case that Y is a surface

Here we use the notation given in 3.10 on page 25 and consider the case where Y is of dimension 2. The base is normal and by corollary 2.7 on page 12 Y is rational. The generic fiber is 1-dimensional, smooth and is therefore isomorphic to a $\mathbb{P}_1$. Using elementary properties of the extremal contractions, one can even see that

LEMMA 5.1. *All the ϕ-fibers are of dimension 1.*

PROOF. If X_μ were not 1-dimensional, then $\dim X_\mu = 2$. Take a curve $C \subset Y$ so that $\mu \in C$. Set $D := \overline{\phi^{-1}(C \setminus \mu)}$. The divisor D intersects an irreducible component of X_μ. Now take a curve $R \subset X_\mu$ intersecting D in finitely many points. We have $R.D > 0$. However, all generic X_η are homologeously equivalent to R (up to positive multiples). So $X_\eta.D > 0$, contradicting the definition of D.

□

In the first section we discuss very special action of groups which are isomorphic to $\mathbb{C}^*$. The result enables us to show that ϕ gives X the structure of a $\mathbb{P}_1$ bundle over Y. In particular both X and Y are smooth. The rest of this chapter is devoted to a further investigation of the birational geometry of X. One cannot expect X to be a simple compactification of a line bundle. However, we are able to show that if G is solvable, then X can be equivariantly transformed into only blow-ups and –downs with centers being smooth curves. This construction will be described explicitly.

The central object in our discussion will be the "rational section", which is in essence a generalization of the notion of a section in a bundle.

DEFINITION 5.2. If $\phi : X \to Y$ is a morphism and $U \in Y$ a dense subset such that ϕ induces a bundle structure on U, Z a subvariety of X having the same dimension as Y and if, for a general point $y \in Y$ we have $Z.\phi^{-1}(y) = 1$, then Z is called a "rational section" over Y.

REMARK 5.3. Let X be a bundle over Y with 1-dimensional fibers. Then a rational section Z is a section if and only if Z contains no ϕ-fibers.

1. Special $\mathbb{C}^*$-actions

Throughout this section, let H^* be a 1-dimensional subgroup of G such that $H^* \cong \mathbb{C}^*$ and H^* does not act on Y.

REMARK 5.4. Take an equivariant resolution $\pi : \tilde{X} \to X$ of the singularities of X. We will furthermore assume that the π-exceptional set is of pure dimension 2 and the the irreducible components are smooth and intersect transversally. Consider $D'_{\tilde{X}} := \mathrm{Fix}(H^*_\eta) \subset \tilde{X}$ and let $D_{\tilde{X}}$ be the union of those irreducible components of

$D'_{\tilde{X}}$ which are not mapped to curves or points by ϕ. Set $D_X := \pi(D_{\tilde{X}})$. The subvariety $D_{\tilde{X}}$ intersects every generic ϕ-fiber at least once. So $D_{\tilde{X}}$ is a divisor.

LEMMA 5.5. *The divisor $D_{\tilde{X}}$ is smooth.*

PROOF. In a linear representation, fixed point sets of groups are vector subspaces. Compact groups can be linearized at the fixed point sets. Thus the fixed point sets of a compact group (or a reductive group) are smooth. □

LEMMA 5.6. *Let $D_{\tilde{X}}$ be defined as above. The set*

$$M := \{y \in Y : \#(\pi^{-1}\phi^{-1}(y) \cap D_{\tilde{X}}) = 1\}$$

is discrete.

PROOF. Linearization of the H^*-action yields that for any point $f \in D_{\tilde{X}}$, there is a unique H^*-stable curve intersecting $D_{\tilde{X}}$ at f. The intersection number is then one.

Assume $\dim M = 1$ and let y be a generic point in M. Because all the components of the π-exceptional divisors are smooth, $\pi^{-1}\phi^{-1}(y)$ contains a smooth curve C as an irreducible component intersecting $D_{\tilde{X}}$. Now $C.D_{\tilde{X}} = 1$, and because $C \cap D_{\tilde{X}}$ was the only intersection point by assumption, $\pi^{-1}\phi^{-1}(y).D_{\tilde{X}} = 1$, contradicting $D_{\tilde{X}}$ intersecting the generic $\phi \circ \pi$-fiber twice. □

LEMMA 5.7. *Let $D_{\tilde{X}}$ and D_X be defined as above. Set $N := \{\mu \in Y \,|\, \dim(X_\mu \cap D_X) > 0\}$. Then N is finite.*

PROOF. Suppose to the contrary that N contained a maximal curve C. C might not be reducible. $\phi^{-1}(C)$ contains at least one irreducible component of D_X which is a contradiction to the construction of $D_{\tilde{X}}$ □

LEMMA 5.8. *The divisor D_X consists of 2 distinct irreducible components.*

PROOF. Let us prove the theorem first in the special case that Y is smooth. By lemma 5.6, $D_{\tilde{X}}$ is a 2-sheeted cover over $Y \setminus (N \cup M \cup \phi(\mathrm{Sing}(X)))$. The set $(N \cup M \cup \phi(\mathrm{Sing}(X))$ is finite, so Y being smooth implies that $Y \setminus (N \cup M \cup \phi(\mathrm{Sing}(X)))$ is simply connected. Hence $D_{\tilde{X}}$ has two connected components over $Y \setminus (N \cup M \cup \phi(\mathrm{Sing}(X)))$. The set $D_{\tilde{X}} \cap \pi^{-1}\phi^{-1}(N \cup M \cup \phi(\mathrm{Sing}(X)))$, however, is just a curve. Therefore $D_{\tilde{X}}$ cannot be irreducible.

There still remains the case that Y is not smooth. We can equivariantly desingularize Y to $\tilde{Y}$ and then find an equivariant desingularization $\tilde{X}$ over X dominating $\tilde{Y}$, i.e. we have a commuting diagram.

$$\begin{array}{ccc} X & \longleftarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Y & \longleftarrow & \tilde{Y} \end{array}$$

We use the result from the last paragraph in order to show that $D_{\tilde{X}}$ has two irreducible components. So its image, D_X, is reducible as well. □

2. The $\mathbb{P}_1$-bundle structure of X

LEMMA 5.9. *Let X and G be as above. Then there exists a rational section $E_1 \subset X$. If G is solvable, then we can choose E_1 to be the compactification of a G-orbit.*

PROOF. We distinguish between the following cases:

Ω contains ϕ-fibers: In this case G cannot be solvable. We take a LEVI-MALCEV decomposition $G = R \rtimes S$ and consider the action of S.

If there is a subgroup $SL_2 < \mathrm{Ker}(\phi)$, then we can easily find an algebraic subgroup of G which acts almost transitively as well but where the open orbit does not contain fibers. We are now in one of the other cases.

If S acts non-trivially on Y, then by $S_\eta \cong SL_2$, we know that $Y \cong \mathbb{P}_2$, $S \cong SL_3$ and X is the homogeneous flag manifold.

G is unipotent: If this is the case we know that for all $\eta \in \Omega_Y$ (i.e. in the open orbit of Y in Ω) there exists a unique $p \in X_\eta$ such that $E_1 := \overline{G.p}$ is a divisor. Simply take p to be the unique $(G_\eta)^0$ fixed point. The fact that algebraic subgroups of unipotent groups are always connected yields that $G_\eta = (G_\eta)^0$ and hence that E_1 intersects a generic fiber once.

G is solvable and $\dim \mathrm{Ker}(\phi : G \to \mathrm{Aut}(Y)) > 0$: In this case we subdivide again

- **K contains $H^* \cong \mathbb{C}^*$:** We may apply lemma 5.8. Take a component of $D_{\tilde{X}}$ and call the ϕ-image E_1. Note, since the open G-orbit in X does not contain a fiber, one of these components coincides locally with a G-orbit. Let E_1 be this divisor.
- **K is unipotent:** Set $E_1 := \mathrm{Fix}(K)$. We use the fact that unipotent groups have only a single fixed point on $X_\eta \cong \mathbb{P}_1$.Since $K < G$, this set is a compactification of a G-orbit.

None of the above holds: We claim that there exists a 2-dimensional connected algebraic subgroup I which is a semi-direct product of a 1-dimensional torus and a 1-dimensional unipotent subgroup, a generic $\eta \in Y$ and $p \in X_\eta$ so that the closure $E_1 := \overline{I.p}$ is a rational section and G-stable.

If Y is $\mathbb{P}_2$ and G either acts transitively or has just 2 orbits, then there is a 3-dimensional semi-simple subgroup S in G so that the Borel subgroup $I := B_S$ has the desired properties. Otherwise, after blowing up a fixed point if necessary, it is enough to consider the case where Y is a HIRZEBRUCH surface and $G = T.U$ is solvable.

Now by assumption G is not unipotent and there is no group $H^* \cong \mathbb{C}^*$ contained in K. Consequently, the maximal torus T of G acts non-trivially on Y. If $\phi(G) \cong \mathbb{C}^* \times \mathbb{C}^*$, then we have to have a non-trivial kernel, which by assumption is also not the case. So T does not act almost transitively on Y. Using the structure theorems for solvable groups and their unipotent parts, we can always find a normal 1-dimensional unipotent subgroup R_1 acting non trivially on Y and a 1-dimensional torus having the same property such that $I := R_1.T_1$ is the group we are looking for.

Let Γ be the finite cyclic isotropy $\Gamma := I_\eta$ at a point of the open I-orbit in Y and let $p_1, p_2 \in X_\eta$ be two Γ fixed points. Note, since the open G-orbit in X does not contain a fiber, one of these I-orbits coincides locally with a G-orbit. Let E_1 be this divisor.

□

PROPOSITION 5.10. *Let X and G be as above. Then X is a $\mathbb{P}_1$-bundle over Y. In particular, both X and Y are smooth.*

PROOF. In order to apply theorem 4.17 on page 38 similarly to as we did in theorem 4.18 on page 39. We have to show that there exists a divisor E_1 intersecting the generic fiber once. This has already been done in the lemma 5.9. □

3. Birational Transformations of X, Part 1

We construct two types of birational transformations of conic bundles. These have been indicated by SARKISOV in [**Sar81**]. We prefer to give an independent proof in our context. We will describe them in two different sections of this chapter, starting with the simpler one:

3.1. The Blow-Up of a ϕ-fiber. After blowing up of a fiber, it might not be a priori clear that we obtain a $\mathbb{P}_1$-bundle again.

PROPOSITION 5.11. *Let X be an almost homogeneous 3-dimensional projective variety, Y an almost homogeneous smooth projective surface and suppose $\phi : X \to Y$ gives X the structure of a $\mathbb{P}_1$-bundle. Let $\mu \in Y$ be G-stable. Then there is a commutative diagram of blow-ups:*

$$\begin{array}{ccc} X_1 & \xrightarrow{\pi_X} & X \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi} \\ Y_1 & \xrightarrow{\pi_Y} & Y \end{array}$$

where π_Y is the blow-up of μ, π_X the blow-up of X_μ and $\phi_1 : X_1 \to Y_1$ is a $\mathbb{P}_1$ bundle.

PROOF. The map ϕ_1 is given by the universal property of the blow-up. The generic ϕ_1 fiber intersects K_{X_1} negatively, because X and X_1 are isomorphic in a neighborhood of the generic fiber. For this reasons, we can do a relative minimal model program over Y_1. Looking at the PICARD number, we can easily show that there can be one relative contraction only and hence that ϕ_1 is indeed a MORI map. By what we have shown in proposition 5.10, X_1 is a $\mathbb{P}_1$-bundle over Y_1. □

4. Rational Sections

Here we investigate the bundle structure of X. Examples show that we cannot generally expect X to be a simple compactification of a line bundle. The full flag manifold $F_{1,2}(3)$ is an example. In fact, X does not even have to have a section.

In the case that G is solvable we will show that after finitely many well-understood transformations, namely the equivariant blow-up and blow-down with center being smooth curves, we can transform X into the compactification of a line bundle over a rational surface $\tilde{Y}$ which is a blow-up of Y. For the rest of this chapter that G is solvable.

LEMMA 5.12. *There exist rational sections E_1 and E_2 in X such that E_1 and $E_1 \cap E_2$ are G-stable*

PROOF. In lemma 5.9 we have already constructed E_1. In order to construct the second rational section, we need to consider a mapping $\pi : Y \to \mathbb{P}_1$. If $Y \cong \Sigma_n$, or a blow-up, there is no problem. If $Y \cong \mathbb{P}_2$, we note that G, by solvabilitiy of G and BOREL's fixed point theorem, there exists a G-fixed point $y \in Y$. By proposition 5.11, we can blow up y and X_y in order to obtain a new $\mathbb{P}_1$-bundle over Σ_1. If we are able to construct our rational sections here, then we can simply take their images to be the desired rational section in the variety we started with. So let us assume that $Y \ncong \mathbb{P}_2$.

Pick a generic point $F \in \mathbb{P}_1$ and set $F_Y := \pi^{-1}(F)$, $F_X := \phi^{-1}(F_Y)$. F_Y is isomorphic to $\mathbb{P}_1$, F_X to a HIRZEBRUCH surface Σ_n. Obviously, G_F acts almost transitively on F_X, stabilizing the curve $E_1 \cap F_X$. Take $x \in F_X$ generic. We decompose G_F into its unipotent part and the maximal torus: $G = R_U \rtimes T$. Now either

R_U acts on F_Y: Take a 1-parameter group $R_1 < R_U$ acting on F_Y. Set $\sigma := \overline{R_1.x}$.

R_U does not act on F_Y: Take a 1-parameter group $H_1^* < T$ acting on F_Y. Set $\sigma := \overline{H_1^*.x}$.

Since neither $H^*_{1\eta}$ nor $R_{1\eta}$ acts on the ϕ-fibers, σ is indeed a section of $F_X \to F_Y$.

We claim that if E_1 and σ intersect, they do so over the G_F-fixed points in F_Y, if any. Indeed, by construction, E_1 and σ do not intersect over the R_1 or H_1^*-orbits, respectively. The only thing we have to show in order to prove our claim is therefore that the G_F-orbit in F_Y coincides with the R_1 or H_1^*-orbit, respectively. This, however, is clear:

If we had to choose H_1^*: and there existed a bigger orbit in F_Y, then $\mathrm{Im}(G \to \mathrm{Aut}(F_Y))$ is either $\mathbb{C}$ or SL_2. In any case, there exists a unipotent group acting on $F_Y \cong \mathbb{P}_1$. A contradiction to our assumption.

In case that we chose R_1: there is only one connected group acting with a bigger orbit: SL_2. This, however, is excluded by the assumption that G was solvable.

The next step is to find a one-parameter group R_2 or H_2^*, exactly as we did above, but with non-trivial action on $\mathbb{P}_1$, the base of Y. Then we define the second rational section $E_2 := \overline{R_2.\sigma}$ or $\overline{H_2^*.\sigma}$, respectively.

We still have to show that $E_1 \cap E_2$ is G-stable. By construction, E_1 and E_2 intersect over G-stable curves on Y, if at all. But E_1 is G-stable. So $E_1 \cap E_2$ is stable as claimed. □

NOTATION 5.13. We keep the notation for the rational sections used in lemma 5.12 throughout the chapter. A fixed general ϕ-fiber will always be called R.

LEMMA 5.14. *Let G be a group acting almost transitively on a smooth rational surface Y. Then every irreducible G-stable curve is smooth.*

PROOF. We first consider the case of a HIRZEBRUCH surface Σ_n. Let $\pi : \Sigma_n \to \mathbb{P}_1$ be the natural projection. There are several different cases to consider:

1. There is a 1-dimensional subgroup $P < G$ such that C is fixed under P.
   (a) P acts non-trivially on $\pi(Y)$ and C is a π-fiber.
   (b) P acts trivially. Then either $P = \mathbb{C}^*$ and C is the zero- or infinity section of Σ_n or $P = \mathbb{C}$ and C is the infinity section or a π-fiber.
2. Every 1-dimensional subgroup $P < G$ acts non-trivially on Y. Then either
   (a) There is a $P < G$ such that $P = \mathbb{C}$. Then C is the closure of a projective $\mathbb{C}$-orbit and therefore smooth.
   (b) There is no unipotent subgroup of G. Then $G = \mathbb{C}^* \times \mathbb{C}^*$. So C is the zero- or infinity section of Σ_n or the fiber over one of the G-fixed points of $\pi(Y)$.

If $Y = \mathbb{P}_2$, then we use the LEVI-MALCEV decomposition on $G = R \rtimes S$. R is normal in G, so G stabilizes the set of R-stable curves. Because of that it suffices to show that R stable curves are smooth, provided R is not trial. The flag theorem (cf. [**Hum75**]) allows us the conjugate $R < SL_3 = \mathrm{Aut}(\mathbb{P}_2)$ in order to make R a group in the upper triangular matrices. So after a coordinate change, $\mathbb{P}_2$ can be decomposed into

1. the open orbit
2. linear R-stable curves.
3. a fixed point.

and the claim is proofed (at least in this special case).

We assume next that R is trivial. Then S is not. So either $S = SL_3$, and there is no stable curve, or $S = SL_2$ and there are exactly three S-stable sets: the open orbit, a fixed point and a line in $\mathbb{P}_2$.

The next case to cope with is if Y is a blow-up of its minimal model: $\pi : Y \to Y_m$. G acts on Y_m as well. Take an irreducible G-stable curve $C \subset Y$. Then either

1. C is an irreducible component of a π-fiber and is smooth because π is just a combination of blow-ups.
2. $\pi(C)$ is a G-stable curve. By what we said above, $\pi(C)$ is smooth. Every point of $\pi(C)$ is of multiplicity one. So $\pi|_C : C \to \pi(C)$ is 1:1 and thus an isomorphism.

□

PROPOSITION 5.15. *If $C \subset E_1 \cap E_2$ is a curve which is not contained in a ϕ-fiber, the C is smooth.*

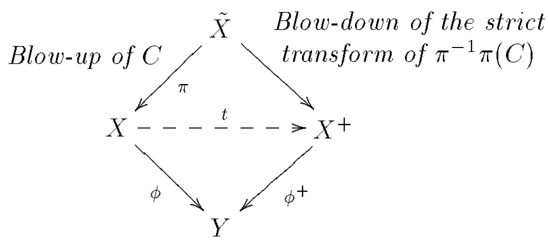
PROOF. Since $E_1 \cap E_2$ is G-stable, its projection $\phi(E_1 \cap E_2)$ is a G-stable curve and thus by lemma 5.14 it is smooth. Apart from finitely many exceptions, E_1 intersects the ϕ-fibers only once. Thus that $\phi|_C$ is generically 1:1 onto its image and it follows that C is smooth as well. □

5. Birational Transformations of X, Part 2

The transformation we now discuss will be referred to as the "elementary". It is similar to those which link the different HIRZEBRUCH surfaces.

5.1. Elementary Transformation. The goal here is to prove the following theorem:

THEOREM 5.16. *Suppose $C \subset E_1 \cap E_2$ is a smooth G-stable curve such that $\phi|_C : C \to \phi(C)$ is injective. Then there is a commutative diagram of equivariant birational transformations:*



where $\phi^+ : X^+ \to Y$ is again the MORI *contraction of a $\mathbb{P}_1$ bundle.*

We call t the "elementary transform" with center C. The elementary transform is equivariant. Before we start working towards a proof, we remind the reader of some notation that will be used throughout this chapter.

NOTATION 5.17.

X	a 3-dimensional projective compact manifold which is almost homogeneous with respect to an algebraic group action of the algebraic group G.
Y	a smooth, compact, rational surface.
$\phi : X \to Y$	a MORI contraction of X to Y which displays X as a $\mathbb{P}_1$-bundle over Y.
R	a general ϕ-fiber.
E_1	a G-stable rational section over Y.
E_2	another rational section over Y. $E_1 \cap E_2$ is G-stable.
C	an irreducible curve in $E_1 \cap E_2$ that is not mapped to a point by ϕ. By proposition 5.15, C is smooth.
H	a very ample divisor on Y.

Now consider $\pi : \tilde{X} \to X$ which is the blow-up of C. If Z is a subvariety of X not contained in C, we denote by $\tilde{Z}$ the π-strict transform of Z. Take x to be a general point of C. Furthermore, set

$$\begin{aligned} R_1 &:= \widetilde{\phi^{-1}\phi(x)} \\ R_2 &:= \pi^{-1}(x) \\ B &:= \pi^{-1}(C) \\ D &:= \widetilde{\phi^{-1}\phi(C)} \\ \hat{H} &:= \pi^{-1}\phi^{-1}(H) \\ K &:= \overline{\{\sum_i a_i C_i | a_i \in \mathbb{R}^+, C_i \text{ a curve in } \tilde{X} \text{ with } \tilde{E}_1.C_i < 0\}} \subset H_2(\tilde{X}, \mathbb{R}) \end{aligned}$$

Figure 1 illustrates what is meant by these objects. Finally we consider the mapping

$$\begin{aligned} \psi : H_2(Y, \mathbb{R}) &\to H_2(\tilde{X}, \mathbb{R}) \\ z &\to \tilde{E}_1 \cap \pi^*\phi^*(z) \end{aligned}$$

We now show that D can be blown down.

LEMMA 5.18. *The projection $\pi : \tilde{X} \to X$ induces a map on cohomology $\pi : H_2(\tilde{X}, \mathbb{R}) \to H_2(X, \mathbb{R})$ such that*

$$\begin{aligned} \pi^{-1}(\mathbb{R}^+[R]) &= (\hat{H})_{=0} \cap \overline{NE(\tilde{X})} \\ &\subseteq \mathbb{R}[[R_1], [R_2]] \subseteq H_2(\tilde{X}, \mathbb{R}) \end{aligned}$$

Furthermore, $\pi^{-1}(\mathbb{R}^+[R]) \cap \overline{NE(\tilde{X})}$ is extremal.

PROOF. Since $\mathbb{R}^+[R]$ is an extremal ray,by lemma 2.28 on page 15 $\pi^{-1}(\mathbb{R}^+[R]) \cap \overline{NE(\tilde{X})}$ is likewise extremal. Now $\pi : \tilde{X} \to X$ is a blow-down. Thus the kernel of $\pi : H_2(\tilde{X}, \mathbb{R}) \to H_2(X, \mathbb{R})$ is one-dimensional and $\dim \pi^{-1}(\mathbb{R}^+[R]) \leq 2$. But $[R_1]$ and $[R_2]$ are obviously contained in $\pi^{-1}(\mathbb{R}^+[R])$. So they span that space. On the

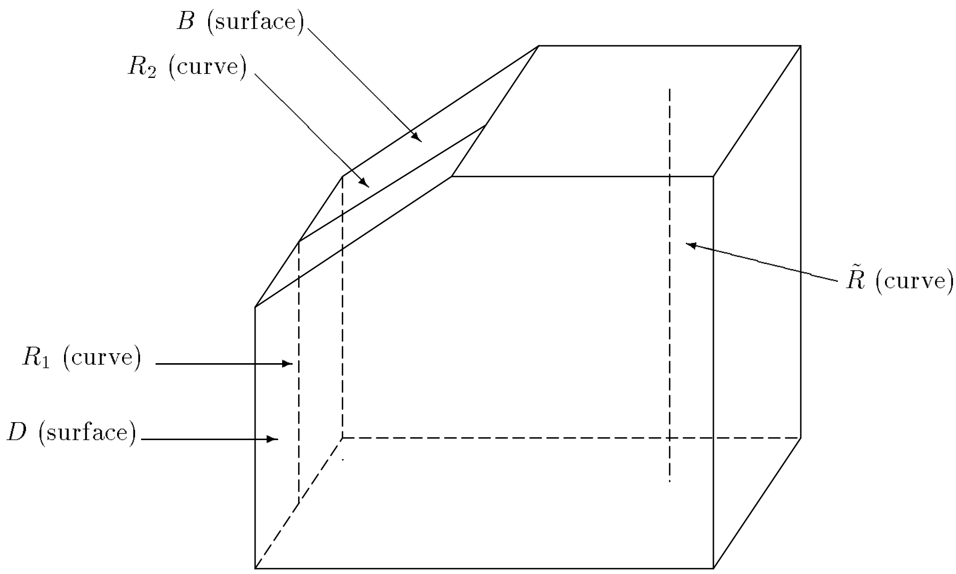


FIGURE 1. The geometry of $\tilde{X}$

other hand, $\dim(H)_{=0} \cap \overline{NE(X)} = 1$ and thus $\dim(\hat{H})_{=0} \cap \overline{NE(\tilde{X})} = 2$. We know that $\hat{H}.R_1 = \hat{H}.R_2 = 0$. So $(\hat{H})_{=0} \cap \overline{NE(\tilde{X})}$ is also spanned by $[R_1]$ and $[R_2]$. □

LEMMA 5.19. *Suppose S is an irreducible curve with $\tilde{E}_1.S < 0$. Then S is contained in $\psi(\overline{NE(Y)})$.*

PROOF. We remark that S is not contained $\tilde{E}_1$. The morphism $\phi \circ \pi$ is 1:1 if restricted to $\tilde{E}_1$. So $\phi \circ \pi$ maps S injectively onto its image in Y. Now we obtain by definition of ψ that $\psi \circ \phi \circ \pi([S]) = [S]$. □

LEMMA 5.20. *The cone K is contained in $\psi(\overline{NE(Y)})$.*

PROOF. Take an element $S \in K$. By definition of K there is a sequence $(C^I)_{I \in \mathbb{N}} \subset K$ such that $C_I = \sum_i a_i^I C_i^I$, where $a_i^I \in \mathbb{R}^+$ and C_i^I curves in $\tilde{X}$ with $\tilde{E}_1.C_i^I < 0$, and $\lim_{I \to \infty}[C^I] = S$. By lemma 5.19, $[C^I] \in \psi(\overline{NE(Y)})$. Since ψ is a linear mapping between finite dimensional vector spaces and is in particular closed, $\psi(\overline{NE(Y)})$ is closed. So $S = \lim_{I \to \infty}[C^I] \in \psi(\overline{NE(Y)})$. □

LEMMA 5.21. *For all $S \in H_2(Y, \mathbb{R})$ it follows that $H.S = \hat{H}.\psi(S)$.*

PROOF. If S is the homology class of a curve, the claim follows from the projection formula (cf. [**Har93**, p. 426]). Since Y is a rational surface, we know that $H_2(Y, \mathbb{R})$ is spanned by classes of complex curves. Since ψ as well as H and $\hat{H}$ (as dual elements of $H_2(Y, \mathbb{R})$ (resp. $H_2(\tilde{X}, \mathbb{R})$) are linear, the claim is proved. □

NOTATION 5.22. The spaces $H_2(\tilde{X}, \mathbb{R})$ and $H_2(Y, \mathbb{R})$ are finite dimensional. For this reason, we can find norm functions on these spaces such that ψ becomes an isometry. Let Δ_Y and $\Delta_{\tilde{X}}$ denote the unit balls, respectively.

LEMMA 5.23. *Define*

$$M := \min_{S \in K \cap \Delta_{\tilde{X}}} \hat{H}.S.$$

Then $M > 0$.

PROOF. This is an immediate consequence of the following:

$$\begin{aligned} M &\geq \min_{S \in \psi(\overline{NE(Y)}) \cap \Delta_{\tilde{X}}} \hat{H}.S && \text{by lemma 5.20} \\ &= \min_{S \in \overline{NE(Y)} \cap \Delta_Y} H.S && \text{by lemma 5.21} \\ &> 0 \end{aligned}$$

The last inequality follows from KLEIMAN's ampleness criterion and the fact that H is ample on Y. □

LEMMA 5.24. *There exists an* $L \in Div(\tilde{X})$ *such that*

1. $\overline{NE(\tilde{X})} \subseteq (L)_{\geq 0}$
2. $L.R_1 = 0$
3. $L.R_2 > 0$

PROOF. If $\overline{NE(\tilde{X})} \subseteq (\tilde{E}_1)_{\geq 0}$, then $L' := \hat{H} + \tilde{E}_1$ already satisfies (1) and (3). If not, set

$$m := \min_{S \in K \cap \Delta_{\tilde{X}}} \tilde{E}_1.S$$

and

$$L' := \lceil -\frac{m}{M} + 10 \rceil \hat{H} + \tilde{E}_1.$$

Then L' fulfills (1) and (3) in any case. If $L'.R_1 = 0$, we are finished. If not, set

$$L'' := L' + (L.R_1)D.$$

Since $D.R_1 = -1$, we have $L''.R_1 = 0$. If (1) still holds, we can stop. If not, we have to modify L'' in a way that it becomes nef and (2) and (3) are not disturbed. Note that D is a HIRZEBRUCH surface. So any curve on D is a linear combination of the 0-section σ_0^D and the fiber R_1. If S is a curve intersecting L'' negatively, then $S \subset D$. For this reason it is sufficient to provide a modification intersecting σ_0^D non-negatively. In short,

$$L := L'' - (\sigma_0^D.L'')\hat{H}$$

satisfies (1)–(3). □

Now we are in a position to describe $\mathrm{Ker}(\hat{H})$ as a part of $\overline{NE(\tilde{X})}$.

PROPOSITION 5.25. *Non-negative linear combinations of* R_1 *and* R_2 *span an extremal subcone of* $\overline{NE(\tilde{X})}$ *which is given by* $(\hat{H})_{=0} \cap \overline{NE(\tilde{X})}$.

PROOF. We know that $\mathrm{Ker}(\hat{H})$ is spanned by R_1 and R_2 and must investigate the problem of which linear combinations of R_1 and R_2 actually lie in $\overline{NE(\tilde{X})}$. Obviously, non-negative combinations do. If we consider J to be an ample divisor on X, then $\overline{NE(\tilde{X})} \subseteq (\pi^*J)_{\geq 0}$. Furthermore $\pi^*(J).R_1 > 0$, $\pi^*(J).R_2 = 0$. Thus for all $\epsilon > 0$ we have $\pi^*(J).(R_2 - \epsilon R_1) < 0$. So $R_2 - \epsilon R_1 \notin \overline{NE(\tilde{X})}$. By lemma 5.24 the same holds if we exchange R_1 and R_2. Thus only non-negative linear combinations are possible. □

COROLLARY 5.26. *Both* R_1 *and* R_2 *are extremal rays.*

PROPOSITION 5.27. *There is a morphism* $\pi^+ : \tilde{X} \to X^+$ *which is the blow-down of the divisor* D *to a smooth curve.*

PROOF. In order to apply MORI's theory of extremal contractions, we must show that $K_{\tilde{X}}.R_1 < 0$. Since $\tilde{X}$ was chosen to be the blow-up of X, it follows (cf. [**Har93**, p. 188]) that $K_{\tilde{X}} = \pi^*(K_X) + B$. Now a general ϕ fiber does not intersect the curve C (which we blew up to obtain $\tilde{X}$). Therefore $\tilde{R} = \pi^*(R)$ and thus $K_{\tilde{X}}.\tilde{R} = \pi^* K_X.\tilde{R} = K_X.R = -2$. On the other hand, $[\tilde{R}] = [R_1] + [R_2]$, $\pi^* K_X.R_2 = 0$ and $R_2.B = -1$. So finally $K_{\tilde{X}}.R_2 = K_{\tilde{X}}.(\tilde{R} - R_1) = -2 - \pi^* K_X.R_1 - B.R_1 = -2 - (-1) = -1 < 0$! By MORI theory, there exists contraction of the extremal ray R_1: $\pi^+ : \tilde{X} \to X^+$.

Consider now the subvarieties of $\tilde{X}$ which are contracted by π^+. First, note that D is covered by curves which are strict transforms of ϕ-fibers. So D is mapped to a variety of lower dimension. If D was mapped to a point, all curves in D would be homologeously equivalent —up to positive rational factors. However, there are non-equivalent curves in D, namely R_1 and those which are not mapped to a point by $\phi \circ \pi$. So D is mapped to a curve.

The classification of contractions of smooth threefolds (cf. [**Mor82**]) has very few cases. There is only one possibility for a contraction mapping a divisor to a point: π^+ is a simple blow-up. □

PROPOSITION 5.28. X^+ *is again a* $\mathbb{P}_1$*-bundle over* Y*. The natural mapping is realized by a* MORI *contraction*

PROOF. Since π^+ is the contraction associated to an extremal ray and all the positive linear combinations of $[R_1]$ and $[R_2]$ form an extremal subcone of $\overline{NE(\tilde{X})}$, it follows from lemma 2.28 on page 15 that $\pi^+(R_2)$ is again an extremal curve. Since $[R_2]$ and $[\tilde{R}]$ only differ by $[R_1]$, $R^+ := \pi^+(\tilde{R})$ is extremal as well.

In order to show that R^+ can be contracted, we must prove that $K_{X^+}.R^+ < 0$. Note that $K_{\tilde{X}} = {\pi^+}^*(K_{X^+}) + D$. Since $\tilde{R}$ does not intersect D, we have $\tilde{R} = {\pi^+}^{-1}(R^+)$ and $\tilde{R}.D = 0$. Thus

$$\begin{aligned} K_{X^+}.R^+ &= {\pi^+}^* K_{X^+}.{\pi^+}^{-1}(R^+) \\ &= {\pi^+}^* K_{X^+}.\tilde{R} \\ &= ({\pi^+}^* K_{X^+} + D).\tilde{R} \\ &= K_{\tilde{X}}.\tilde{R} \\ &= (\pi^* K_X + B).\pi^{-1} R \\ &= K_X.R \\ &= -2 \end{aligned}$$

So R^+ can be MORI contracted. Now R^+ is just the $\pi {\pi^+}^{-1}$ strict transform of a general ϕ fiber and we find a ZARISKI open subset of X^+ (namely the strict transform of $\phi^{-1}(Y \setminus \phi(C))$) is covered by such curves. So the contraction $\phi^+ : X^+ \to Y^+$ maps X^+ to something of lower dimension. We exclude the cases $\dim Y^+ = 0, 1$.

$\dim \mathbf{Y}^+ = 0$: If this were the case, all curves in X^+ were in the same homology class (as usual up to positive rational factors). Since $R^+.\pi^+(B) = \tilde{R}.(B + D) = 0$ and there do exist curves S intersecting $\pi^+(B)$ properly, $[R^+]$ and $[S]$ cannot be positive multiples of each other.

$\dim \mathbf{Y}^+ = 1$: If this were the case and F were a general (2-dimensional) ϕ^+-fiber, then without loss of generality $R^+ \subset F$. Thus there is a curve $S \subset F$ intersecting R^+ properly. Now $\phi\pi{\pi^+}^{-1}(R^+)$ is a point, but $\phi\pi{\pi^+}^{-1}(S)$ is not. So $[R^+]$ and $[S]$ cannot be positive multiples of each other.

Now we know that Y^+ is a surface. As above, X^+ is a $\mathbb{P}_1$-bundle over Y^+. Recall that E_1 is a rational section in X and that $\phi^+\pi^+|_{\tilde{E}_1} : \tilde{E}_1 \to Y^+$ is injective outside of the set $S := \pi^{-1}\{\phi\text{-fibres in } E_1\}$. So $\phi^+\pi^+|_{\tilde{E}_1 \setminus S} : \tilde{E}_1 \setminus S \to Y^+$ is an isomorphism. The same holds for $\phi\pi|_{\tilde{E}_1 \setminus S} : \tilde{E}_1 \setminus S \to Y$. This yields a birational map $t : Y \to Y^+$ such that the sets of points where t or t^{-1} are not isomorphic are both of codimension 2. A version of ZARISKI's main theorem, as presented in [**Har93**, p. 410], tells us that t has to be a regular isomorphism. So $Y = Y^+$ and X^+ is another a $\mathbb{P}_1$ bundle over Y. □

6. The transformation to the compactification of a line bundle

6.1. Eliminating vertical curves. Let $S \subset \phi(E_1 \cap E_2)$ be an irreducible curve which is a ϕ-fiber. We say that E_1 and E_2 intersect vertically in S. Proposition 5.11 on page 44 ensures that after blowing up S we obtain again a $\mathbb{P}_1$-bundle. We call this transformation $T_0 : X_1 \to X$. The proper transforms of E_1 and E_2 are still rational sections. If they still intersect in a ϕ_1-fiber over $t_0^{-1}\phi(S)$, the blowing-up can be applied again. So we eventually get a sequence of blow-ups such that the following diagram commutes.

$$\begin{array}{ccccccc} X = X_0 & \xleftarrow{T_0} & X_1 & \xleftarrow{T_1} & X_2 & \xleftarrow{T_2} & \cdots \\ \downarrow{\scriptstyle \phi=\phi_0} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \\ Y = Y_0 & \xleftarrow{t_0} & Y_1 & \xleftarrow{t_1} & Y_2 & \xleftarrow{t_2} & \cdots \end{array} \tag{5.1}$$

The strict transforms of the E_1 and E_2 are again rational sections in X_i. We denote them by E_1^i or E_2^i, respectively.

PROPOSITION 5.29. *The sequence (5.1) terminates, i.e. there exists a number $i \in \mathbb{N}$ such that the strict transforms E_1^i and E_2^i do not intersect vertically.*

PROOF. If $S =: S^{(0)}$ is given and $s \in S$ a generic point, we can find a local section of the bundle X containing s. Let $U \subset X$ be that section. By general choice of s, $E_1 \cap U \neq E_2 \cap U$. We know by [**Hir62**] that we can resolve the singularities of $(E_1 \cup E_2) \cap U$ by repeatedly blowing up the intersection point. Now $U^1 := T_0^{-1}$ is just the blow-up of U at a point in $E_1 \cap E_2 \cap U$.

Furthermore, $(E_1^{(1)} \cup E_2^{(1)}) \cap U^{(1)}$ is the strict transform of $(E_1 \cup E_2) \cap U$ under the blow-up of U. Suppose for a moment that $E_1^{(1)} \cap U^{(1)}$ and $E_2^{(1)} \cap U^{(1)}$ were disjoint. If so, $E_1^{(1)}$ and $E_2^{(1)}$ would not intersect over any point in $t_0^{-1}\phi(S)$. If they are not disjoint, by general choice of U, $E_1^{(1)} \cap U^{(1)}$ and $E_2^{(1)} \cap U^{(1)}$ do not contain a T_0-exceptional curve. So we may continue with our process.

However, once that the singularities of $(E_1^{(i)} \cup E_2^{(i)}) \cap U^{(i)}$ are resolved, the process stops. So $E_1^{(i)}$ and $E_2^{(i)}$ do no longer intersect in ϕ_i-fibers. □

6.2. Eliminating horizontal curves. We may now assume that E_1 and E_2 do not intersect vertically. Let $S \subset \phi(E_1 \cap E_2)$ be an irreducible curve. Then there is a unique curve $C^{(0)} \subset \phi^{-1}(S) \cap E_1 \cap E_2$ giving rise to a uniquely defined birational transformation as ensured by theorem 5.16. This transformation is denoted by $T^{(0)} : X \dashrightarrow^{eq} X^{(1)}$. The strict transforms of E_1 and E_2 are again rational sections. If they still intersect over S, we again obtain a curve $C^{(1)}$ over S and have another transformation. So we continue the process and obtain a sequence of transformations such that the following diagram commutes.

$$X = X^{(0)} \overset{T^{(0)}}{\dashrightarrow} X^{(1)} \overset{T^{(1)}}{\dashrightarrow} X^{(2)} \overset{T^{(2)}}{\dashrightarrow} \cdots \qquad \phi = \phi^{(0)} : X^{(0)} \to Y, \quad \phi^{(1)} : X^{(1)} \to Y, \quad \phi^{(2)} : X^{(2)} \to Y \tag{5.2}$$

The main theorem of this section is

THEOREM 5.30. *The sequence (5.2) terminates after finitely many transformations, i.e. there exists a $j \in \mathbb{N}$ such that for all curves $C \in E_1^{(j)} \cap E_2^{(j)}$ it follows that $\phi^{(j)}(C) \neq S$. Furthermore, if E_1 and E_2 do not intersect vertically, then $E_1^{(i)}$ and $E_2^{(i)}$ do not intersect vertically for all i.*

PROOF. Take a generic smooth rational curve $Q_Y \subset Y$ intersecting S properly in exactly one point y. We write $Q^{(i)} := {\phi^{(i)}}^{-1}(Q_Y)$. For ease of notation we may assume without loss of generality that $R_1^{(i)}$ and $R_2^{(i)}$ are contained in $\widetilde{Q^{(i)}}$. All the $Q^{(i)}$ are $\mathbb{P}_1$-bundles over Q_Y, thus HIRZEBRUCH surfaces. By generic choice of Q_Y, the $E_1^{(i)} \cap Q^{(i)}$ and $E_2^{(i)} \cap Q^{(i)}$ are sections of $Q^{(i)}$.

We want to investigate how $Q^{(i)}$ and $Q^{(i+1)}$ are related to each other. Blowing up the curve $C^{(i)}$, the strict transform of $Q^{(i)}$ (let's call it $\widetilde{Q^{(i)}}$) is the blow-up of $Q^{(i)}$ at the point $C \cap Q^{(i)}$ (cf. [**Har93**]). But $C \cap Q^{(i)}$ is exactly the intersection point of $E_1^{(i)}$ and $E_2^{(i)}$ in $Q^{(i)}$. Similarly we obtain that $Q^{(i+1)}$ is the blow down of $R_1^{(i)} \subset \widetilde{Q^{(i)}}$ to a point.

Suppose for a moment that $\widetilde{E_1^{(i)}} \cap \widetilde{Q^{(i)}}$ and $\widetilde{E_2^{(i)}} \cap \widetilde{Q^{(i)}}$ were disjoint and do not intersect $R_1^{(i)}$. Then, after blowing down $R_1^{(i)}$, they are still disjoint. But blowing down $R_1^{(i)}$ is exactly what the transformation $T^{(i)}$ does! So at this stage $E_1^{(i+1)}$ and $E_2^{(i+1)}$ do not intersect over S. We want to show that exactly this occurs after finitely many transformations.

From the theory of resolutions of curves embedded in surfaces (cf. [**Har93**]), it follows that $E_1^{(i)} \cap Q^{(i)}$ and $E_2^{(i)} \cap Q^{(i)}$ become disjoint over y if we repeatedly blow up the intersection points that lie over y. Since $E_1^{(i)} \cap Q^{(i)}$ is a section of Q^i, it is smooth and thus for all $x \in E_1^{(i)} \cap Q^{(i)}$ we have for the tangent spaces

$$T_x(E_1^{(i)} \cap Q^{(i)}) \not\subseteq T_x({\phi^{(i)}}^{-1}\phi^{(i)}(x)).$$

So $\widetilde{E_1^{(i)}} \cap R_1^{(i)} = \emptyset$. This leads to the following two possibilities: If $\widetilde{E_1^{(i)}}$ and $\widetilde{E_2^{(i)}}$ are disjoint, we are finished after blowing down $R_1^{(i)}$. If not, $\widetilde{E_1^{(i)}} \cap \widetilde{E_2^{(i)}} \cap R_1^{(i)} = \emptyset$. So prior to blowing up the intersection point again, we can blow down $R_1^{(i)}$ without changing $\widetilde{E_1^{(i)}} \cap \widetilde{Q^{(i)}}$ and $\widetilde{E_2^{(i)}} \cap \widetilde{Q^{(i)}}$ over y in any way. So the process terminates after finitely many transformations.

It still must be shown that $E_1^{(i)}$ and $E_2^{(i)}$ do not intersect vertically. This will be done by proving that if $E_1^{(i)}$ and $E_2^{(i)}$ have non-empty vertical intersection, then $E_1^{(i-1)}$ and $E_2^{(i-1)}$ as well. Since E_1 and E_2 do not, the claim follows. Assume that $V^{(i)} \subset E_1^{(i)} \cap E_2^{(i)}$ is a vertical curve over S. Without loss of generality it may be assumed that $R_2^{(i-1)}$ is the strict transform of $V^{(i)}$ in $X^{(\tilde{i-1})}$ and that $R_1^{(i-1)}$ lies over the same point of S as $R_2^{(i-1)}$ does. The curve $R_1^{(i-1)}$ does intersect the divisor $D^{(i-1)}$. On the other hand, as we have seen above, if we take $D^{(i-1)}$ to be the strict transform of ${\phi^{(i-1)}}^{-1}(S)$ in $\widetilde{X^{(i-1)}}$, then $\widetilde{E_1^{(i-1)}}$ and $D^{(i-1)}$ do not intersect over generic points of S. Since $\widetilde{E_1^{(i-1)}}$ cannot intersect $D^{(i-1)}$ in finitely many point only, it follows that $R_1^{(i-1)} \subset \widetilde{E_1^{(i-1)}}$. The same holds for $\widetilde{E_2^{(i-1)}}$, so that $E_1^{(i-1)}$ and $E_2^{(i-1)}$ indeed intersect vertically.

□

6.3. The construction of independent sections. By proposition 5.29 the variety X can be transformed into a $\mathbb{P}_1$ bundle such that the strict transforms of E_1 and E_2 do not intersect in fibers. A second transformation will rid us of curves in $E_1 \cap E_2$ which are not contained in fibers. Since the latter transformation does not create new curves in the intersection, the strict transforms of E_1 and E_2 eventually become disjoint. The resulting space is the compactification of a line bundle.

LEMMA 5.31. *If E_1 and E_2 do not intersect, X is the compactification of a line bundle.*

PROOF. Since E_1 and E_2 are disjoint, neither contains a fiber thus they are sections. □

As a net result, we state

PROPOSITION 5.32. *If X is a linear $\mathbb{P}_1$ bundle over a surface Y, almost homogeneous with respect to a linear solvable group, then there is an algorithmic way of finding a sequence of G-equivariant elementary transformations and blowing up fibers such that the transformed variety is the compactification of a line bundle over an equivariant blow-up of the surface Y.*

Part 3

Birational Classification

CHAPTER 6

Equivariant Rational Fibrations

PROPOSITION 6.1. *Let X be a projective 3-dimensional variety and which is almost homogeneous with respect to the algebraic action of a linear algebraic group G. Then either*

1. *G is reductive or*
2. *there exists an equivariant rational map $X \dashrightarrow^{eq} Y$, where $\dim Y < 3$ or*
3. *there exists an equivariant rational map $X \dashrightarrow^{eq} \mathbb{P}_3$.*

PROOF. Let $G = U \rtimes L$ be the LEVI decomposition of G, i.e. U is unipotent and L reductive and define A to be the center of U. Note that A is non-trivial. Since A is completely canonically defined, it is normalized by L, hence it is normal in G. Let H be the isotropy group of a generic point, so that $\Omega \cong G/H$, and consider the map

$$\Omega \cong G/H \to G/(A.H)$$

There are two things to note. The first is that A is not contained in H (or else G acted with positive dimensional ineffectivity). So $\dim G/(A.H) < 3$. If $\dim G/(A.H) > 0$, it can always be equivariantly compactified $G/(A.H)$ to a variety X' yielding an equivariant rational map $X \dashrightarrow X'$. This is (2) of the claim.

If $\dim G/(A.H) = 0$, then A acts transitively on Ω. In this case $A \cong \mathbb{C}^n$, and hence (because the G-action is algebraic) $\Omega \cong \mathbb{C}^3$.

The theorem on MOSTOW fibration (see [**Mos55b**] and [**Mos55a**] or [**Hei91**, p. 641]) yields that L has to have a fixed point in Ω. Therefore, without loss of generality, $L < H$. As a next step, consider the group $B := (U \cap H)^0$. Since both U and H are normalized by L, B is as well. Elements in A commute with all elements of U, hence $A.B$ normalizes B as well. Note that $A.B = U$, because $A.B = A.(H \cap U) = (A.H) \cap U = G \cap U = U$. Then B is a normal subgroup of $U \rtimes L = G$. Consequently B does acts trivially and so it is trivial.

We are now in a position where we may write $G = A \rtimes_\rho L$, where ρ is the action of L on A (L acting by conjugation). Now $H = L$, hence $A \cong \Omega \cong \mathbb{C}^3$ and the L-action on $A \cong (\mathbb{C}^3, +)$ has to be linear. So G is a subgroup of the affine group and Ω can be equivariantly compactified to $\mathbb{P}_3$, yielding an equivariant rational map $X \dashrightarrow^{eq} \mathbb{P}_3$. □

We now study the case (1) of the preceding proposition in more detail.

PROPOSITION 6.2. *Let X be as above and assume that G be reductive. Assume furthermore that G is not semisimple. Then there is an equivariant rational map $X \dashrightarrow^{eq} Z$, where $Z \cong \mathbb{P}_3$ or $\dim Z = 2$.*

PROOF. As a first step, recall that $G = T.S$, where S semisimple, T a torus, and S and T commute and have only finite intersection. If η is a point in the open orbit and G_η the associated isotropy group, then $T \not\subset G_\eta$, or otherwise T would not act at all. For that reason we will be able to find a 1-parameter group $T_1 < T$, $T_1 \not\subset G_\eta$ and consider the map

$$\Omega := G/G_\eta \to G/(T_1.G_\eta).$$

Since T_1 has non-trivial orbits, $\dim G/(T_1.G_\eta) = 2$. If we compactify the latter in an equivariant way to a variety Z, we automatically obtain a an equivariant rational map $X \dashrightarrow^{eq} Z$ as claimed.

□

LEMMA 6.3. *Suppose G is semisimple. Then one of the following holds:*

1. *$G \cong SL_2$ and the open orbit Ω is isomorphic to SL_2/Γ, where Γ is discrete.*
2. *X is isomorphic to $\mathbb{P}_3$*
3. *X is isomorphic to $F_{1,2}(3)$, is the full flag variety*
4. *X is homogeneous and isomorphic to Q_3, the 3-dimensional quadric.*
5. *X admits an equivariant rational map onto a variety of dimension < 3. The group G acts transitively on the image space.*

PROOF. Let us assume for the rest of this proof that $G \not\cong SL_2$. If G acts transitively, then the we have only few possibilities:

$X \cong \mathbb{P}_3$: This is possible.
$X \cong Q_3$: Again, this is possible and included in the lemma.
$X \cong F_{1,2}(3)$: where $F_{1,2}(3)$ is the full flag variety.
X is a torus-principal bundle: over a homogeneous rational manifold. This case does not occur because G acts algebraically.

Next we assume that G does not act transitively. By blowing up lower dimensional components of the G-exceptional set, we obtain a new variety $\tilde{X}$ with exceptional set $\tilde{E}$, where all components of $\tilde{E}$ are smooth divisors. Note that $\tilde{E}$ does not contain a G-fixed point: linearization at this point would imply G not acting almost transitively. If $\tilde{E}$ contains a G-stable curve C, then there is a map $G \to \mathrm{Aut}(C) \cong SL_2$. The kernel of this map again semi-simple, fixes every point of C and stabilizes $\tilde{E}$. This cannot happen unless the kernel is trivial and $G \cong SL_2$ which is ruled out by assumption. If $\tilde{E}$ does not contain G-stable curves, then the components of $\tilde{E}$ are smooth G-homogeneous surfaces and again there is no kernel. Hence $G \cong SL_3$ or $SL_2 \times SL_2$.

Because we know all the possibilities for the components of $\tilde{E}$, we know that a maximal compact subgroup $K < G$ acts transitively on the components of $\tilde{E}$. The slice theorem, applied to the action of K, yields that the K-orbits in Ω are real-analytic hypersurfaces. Thus, if the G-isotropy of a generic point in Ω is reductive, then Ω is isomorphic to the tangent bundle of a symmetric space of rank 1, equipped with its standard invariant structure (see [**MN63**] for a proof). The only possibilities for Ω are the affine quadric and the complement of a non-degenerate quadric in $\mathbb{P}_3$ —these are the tangent bundles of the 3-sphere and the 3-dimensional projective space respectively. Note that if we have two G-equivariant compactifications of the same homogeneous space and G acts transitively on the components of the exceptional sets, then since the indeterminacy locus of the associated equivariant birational map is G-stable, if E is of pure codimension 1, then the map is biregular, Since the above open orbits are affine, it follows that the obvious compactifications are the unique ones.

The other possibility is that the G-isotropy H is not reductive. Recall that maximal subgroups of algebraic groups are either parabolic or reductive. in our case, we find a minimal parabolic subgroup P containing H. Remember that G/P is a rational compact homogeneous variety. There exists an equivariant rational map $X \dashrightarrow^{eq} G/P$ as claimed. $\square$

Remark 6.4. The case of lemma 6.3 is in fact very well investigated —see e.g. [**HAR85**]. See also theorem 7.14 on page 67 for more complete results.

CHAPTER 7

Linkage to Minimal Models

Recall that in chapter 6 we found that in all relevant cases equivariant (bi)rational mappings $X \dashrightarrow^{eq} Z$ exist, where $0 < \dim Z < 3$, or $Z \cong \mathbb{P}_3$. If $\dim Z < 3$, resolving the maps and performing a relative minimal model program over Z gives then rise to equivariant mappings $X \dashrightarrow X^{(m)}$, where $X^{(m)}$ is one of the minimal models we have discussed so far. The key point of this chapter is that, by choosing the $X^{(m)}$ carefully, one can assume that $X^{(m)}$ admits a $\mathbb{P}_1$-bundle structure or G has no fixed points. In both cases, after blowing up X and $X^{(m)}$, if necessary, the map factors into a sequence of blow-downs. The main purpose here is to construct these modifications.

We discuss the different possibilities for Z separately.

1. Rational Mappings to $\mathbb{P}_3$

LEMMA 7.1. *Let $\phi : X \to Y$ be a birational morphism between smooth projective varieties. If $C \subset Y$ is a curve in $T(\phi^{-1})$, the fundamental points of ϕ^{-1}, then there exists a curve $C' \subset X$ with $C'.K_X < 0$ and $\phi(C')$ a point in C. In particular, ϕ factors through a relative* MORI *contraction over Y.*

PROOF. Let $L \in \mathrm{Pic}(Y)$ be very ample, and $D \in |L|$ be a generic element of the linear system, hence smooth. Note that D intersects C transversally. The divisor $\phi^{-1}(D)$ is a generic element in $|\phi^*(L)|$. So $\phi|_{\phi^{-1}(D)} : \phi^{-1}(D) \to D$ is a birational morphism between smooth surfaces and factors into a sequence of blow-downs. Let C' be a exceptional curve of first type in $\phi^{-1}(D)$. The claim follows from $C.K_{\phi^{-1}(D)} < 0$, the adjunction formula and $C'.\phi^{-1}(D) = 0$. □

LEMMA 7.2. *Let $\phi : X \to X'$ be an equivariant birational morphism. Assume that X' does not have a fixed point. Then ϕ factors into a sequence of blow-downs.*

PROOF. Using the lemma 7.1 and the fact that the set of fundamental points of ϕ^{-1} is G-stable curve, we find a relative contraction over $\mathbb{P}_3$. Since G is acting without fixed point, the contraction is divisorial. Thus the contracted divisor is mapped to a curve. The classification of extremal contractions yields that the contraction is actually a simple blow-down and we start anew. □

PROPOSITION 7.3. *Let $X \dashrightarrow^{eq} \mathbb{P}_3$ be an equivariant birational map. Then either X has an equivariant rational fibration with 2-dimensional base variety or X and $\mathbb{P}_3$ are equivariantly linked by a sequence of blowing up X followed by a sequence of blow-downs.*

PROOF. If the G-action on $\mathbb{P}_3$ has a fixed point, we can blow up this point and obtain a map from the blown-up $\mathbb{P}_3$ to $\mathbb{P}_2$. If there is no such G-fixed point in $\mathbb{P}_3$, by proposition 3.5, after replacing X by an equivariant blow-up, there is a regular equivariant map $\phi : X \to \mathbb{P}_3$. Now lemma 7.2 applies. □

2. Rational mappings to $\mathbb{P}_1$

Now we consider the case where X is mapped to $\mathbb{P}_1$.

LEMMA 7.4. *Let $\theta : X \dashrightarrow^{eq} \mathbb{P}_1$ be an equivariant rational map with generically connected fibers. Then there are morphisms*

$$\begin{array}{ccccc} & & \tilde{X} & & \\ & \stackrel{\beta}{\swarrow} & & \searrow & \\ X & & & & Y \\ & & & & \downarrow \alpha \\ & & & & Z \end{array}$$

where β is a sequence of blow-ups, $Z = \mathbb{P}_1$ or a rational surface and α a is MORI-*contraction which realizes $Y \to Z$ as a bundle.*

PROOF. We blow up X to $\tilde{X}$ in order to desingularize it and to regularize the map to $\mathbb{P}_1$. Next we perform a relative minimal model program over $\mathbb{P}_1$. This program has to terminate, i.e. there will be a dimension reducing contraction at the end. Because this contraction is relative over $\mathbb{P}_1$, Z cannot be a point. □

LEMMA 7.5. *Suppose $\pi : Y \to \mathbb{P}_1$ is a $\mathbb{P}_2$-bundle and there exists a G-stable section T. Then there exists a diagram*

$$\begin{array}{ccc} Y & \stackrel{\iota}{\longleftarrow} & Y' \\ \pi \downarrow & & \downarrow \pi' \\ \mathbb{P}_1 & & Z \end{array}$$

where ι is the blow-up of T and Y' is a $\mathbb{P}_1$-bundle over Z.

PROOF. It is only necessary to construct π'. The $\pi \circ \iota$-fibers are isomorphic to the HIRZEBRUCH surface Σ_1. By the adjunction formula, all curves contained in $\pi \circ \iota_y$-fibers intersect the canonical bundle $K_{Y'}$ negatively. Hence there are two contractible extremal rays in the fiber. One of them is associated to the blow-down to T. We contract the other and obtain a map $\pi' : Y' \to Z$. Because this is a relative contraction over $\mathbb{P}_1$, we find a map $\iota_Z : Z \mapsto \mathbb{P}_1$. In order to show that Y' is a $\mathbb{P}_1$-bundle over Z, by the results from chapter 5 it is only necessary that Z is a surface.

The mapping π' maps $\pi \circ \iota$-fibers to ι_Z-fibers. In particular, a ι_Z-fiber must be an equivariant image of Σ_1. There are only three possibilities:

- **ι_Z-fibers are points:** This is impossible. The fibers of π' cannot be Σ_1 because this would imply $Z' \cong \mathbb{P}_1$, $\pi' = \pi \circ \iota_y$ and $\rho(Y'/Z') = 2$, the last equality contradicting π' being a MORI-contraction.
- **ι_Z-fibers are isomorphic to $\mathbb{P}_1$:** This is obviously possible.
- **ι_Z-fibers are isomorphic to $\mathbb{P}_2$:** There is only one way to map Σ_1 to $\mathbb{P}_2$ with connected fiber: contract the ∞-section. We have already ruled out this case by choosing the other extremal ray in order to construct π'.

So the only remaining case is that where Z is a surface and Y' a bundle. □

LEMMA 7.6. *Let Y be a G-almost homogeneous linear $\mathbb{P}_2$-bundle over $\mathbb{P}_1$. If there does not exist an equivariant rational map $Y \dashrightarrow^{eq} Z$, where Z is a surface or $Z \cong \mathbb{P}_3$, and G has a fixed point on X, then the ineffectivity of the G-action on the base $\mathbb{P}_1$ contains the non-trivial semisimple part of G and $\mathbb{C}^2$.*

PROOF. Let H be the isotropy group of a general point of the open orbit Ω, and let us assume at first that G is reductive. We decompose G into the semisimple part S and a maximal torus T: $G = S.T$.

If T is not trivial, then take $T_1 < T$ to be a normal 1-dimensional group not contained in H, then $\Omega = G/H \to G/(T_1.H)$ is a map to a surface, inducing an equivariant rational map from Y to an equivariant closure of $G/(T_1.H)$. If T is trivial, then take a generic $\eta \in \mathbb{P}_1$ and check whether G_η has fixed points on the fiber or not. If it has and f is one of them, then $G.f$ is a G-stable section in Y, and can be blown up in order to obtain a $\mathbb{P}_1$-bundle over a surface by lemma 7.5. If G_η does not have a fixed point, then —by G acting transitively on $\mathbb{P}_1$— G does not have any fixed point on Y at all, and we are finished.

Now let us assume for the rest of this proof that G is *not* reductive. Going through the proof of proposition 6.1 (see page 57), we see that there exists an equivariant rational map to a surface or to $\mathbb{P}_3$, unless the A, the commutator of the unipotent radical, is non-trivial and $\Omega \cong G/H \to G/(A.H)$ is a map to a curve. In particular, this implies that the generic A-orbits are 2-dimensional. If the semisimple part of G is trivial, then $G = R_U \rtimes T$. Note that there exists a 1-dimensional subgroup $B < A$ which is stabilized by T, hence normal. Then $G/H \to G/(H.B)$ is a map to a surface.

So let us assume that the semisimple part of S of G is not trivial. Now $G = (R_U \rtimes T) \rtimes S$. If the semisimple part acts on the basis, we are finished as we have seen above. □

PROPOSITION 7.7. *Let Y be a G-almost homogeneous linear $\mathbb{P}_2$-bundle over $\mathbb{P}_1$. Then there exists an equivariant rational map $Y \dashrightarrow^{eq} Z$, where Z is a surface or $Z \cong \mathbb{P}_3$ or there exists a sequence of equivariantly blowing up and down $Y \dashrightarrow^{eq} Y'$, and Y' is a linear $\mathbb{P}_2$-bundle over $\mathbb{P}_1$, and the G-action on Y' is fixpoint-free.*

PROOF. By lemma 7.6 we may assume that S, the semisimple part of G, does not act on the base and that the S-action on the fibers has a unique fixed point. Let C be the curve of the S-fixed points. Again, we assume that C is not G-stable, or else we blow up C and obtain a $\mathbb{P}_1$-bundle over a surface. Suppose that G still has a G-fixed point f. Then $f \in C$, and we construct X', X'' and X''' as shown in figure 1 on the following page.

We claim that $\epsilon(R_1)$ is a contractible extremal curve. Note that ϵ is simply the blow-down of the surface over C'. In particular, X'' and X''' are isomorphic outside of C', or its preimage $\epsilon^{-1}(C')$, so that $K_{X''}.R_1 = K_{X'''}.\epsilon(R_1) < 0$. By lemma 2.28 on page 15, $\epsilon(R_1)$ is extremal, so that we can contract it and obtain $X^{(1)}$, which is a $\mathbb{P}_2$-bundle again.

This way we have constructed an equivariant birational transformation which we will now use in order to remove the G-fixed points. Let $g \in G$ be an element not stabilizing C. The curves gC and C meet in f. We know that after finitely many blow-ups of the intersection points of C and gC, the curves become disjoint, so that there no longer exists a G-fixed point! This, however is exactly what we do when applying our transformation.

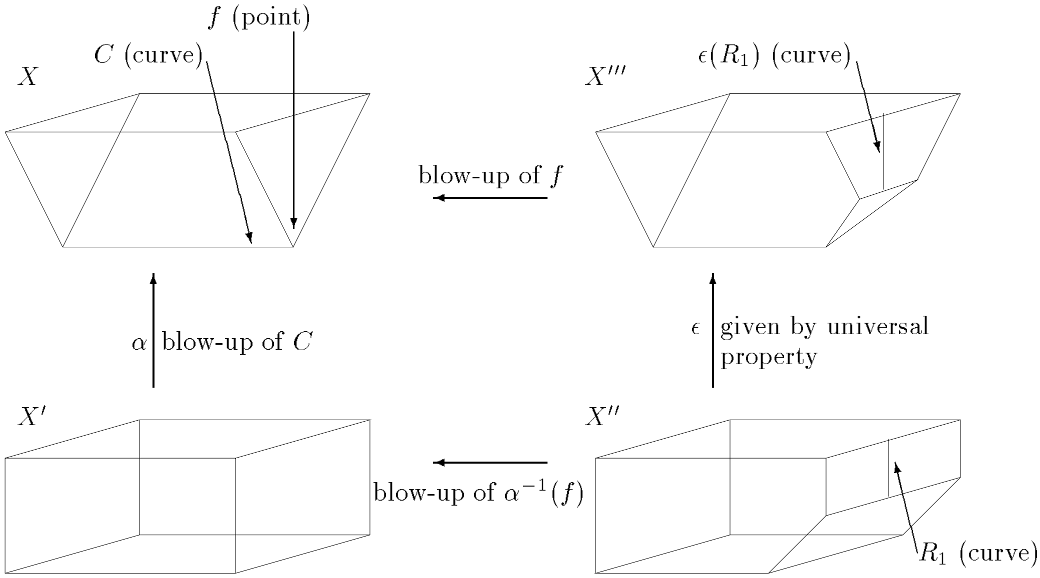


FIGURE 1. Transformation of a $\mathbb{P}_2$-Bundle

□

REMARK 7.8. We could use lemma 7.6 in order to find a complete description of the situation.

2.1. The Blow-Down to a $\mathbb{P}_1$-Bundle. In this section we consider the situation where there is a sequence of morphisms

$$X \xrightarrow{\phi} Y \xrightarrow{\pi} Z, \qquad \theta = \pi \circ \phi : X \to Z$$

with Y and Z being smooth varieties, ϕ a birational morphism and $\pi : Y \to Z$ a $\mathbb{P}_1$-bundle. Furthermore, since the fibers of all these maps are connected, all three varieties are almost homogeneous. In particular, Z is a rational surface.

LEMMA 7.9. *Operating under the assumptions as above, the varieties X, Y and Z can be equivariantly blown up in order to obtain the following diagram*

$$\begin{array}{ccc} X & \xleftarrow{\iota_X} & X' \\ \downarrow \phi & & \downarrow \phi' \\ Y & \xleftarrow{\iota_Y} & Y' \\ \downarrow \pi & & \downarrow \pi' \\ Z & \xleftarrow{\iota_Z} & Z' \end{array} \qquad \theta = \pi\circ\phi,\quad \theta' = \pi'\circ\phi'$$

where $\pi' : Y' \to Z'$ is again a $\mathbb{P}_1$-bundle, the ι_X, ι_Y and ι_Z are equivariant blow-ups and ${\theta'}^{-1}(z)$ does not contain a divisor for all $z \in Z$.

PROOF. Let $\eta \in Z$ be a point such that $\theta^{-1}(\eta)$ contains a divisor. We can decompose $\theta^{-1}(\eta)$ into irreducible parts:

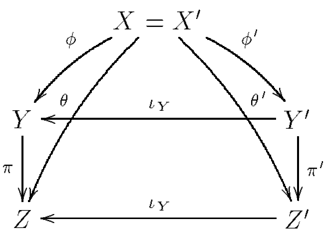
$$\theta^{-1}(\eta) = \bigcup_i D_i \cup \bigcup_j C_j$$

where the C_j are curves and the D_i divisors. Since η is a G-fixed point, the singularities of the C_j are also fixed. Because they are fixed points, we can equivariantly resolve them. Hence we may assume without loss of generality that the C_j are smooth.

The next step is to blow up the C_j, provided they exist. We do this in an arbitrary order, creating new divisors E_j. The last step is to blow up $\pi^{-1}(\eta)$ in order to obtain Y'. We know by proposition 5.11 on page 44 that Y' is again a $\mathbb{P}_1$-bundle over a blow-up of Z. The universal property of the blow-up guarantees the existence of ϕ'.

In order to show that our process improves the situation, we still must show that $I_\theta := \#D_i$, the number of divisors contained in a fibers, strictly decreases: $I_{\theta'} < I_\theta$. We consider the following two cases:

$\theta^{-1}(\eta)$ is of pure dimension 2: If this is the case, we have created a new diagram:



Since every θ'-exceptional divisor is also θ-exceptional, $I_{\theta'} \leq I_\theta$. Because the generic θ'-fiber is 1-dimensional, and the fiber-dimension is semi-continuous, there exists an i such that $\theta'(D_i) = \iota_Z^{-1}(\eta)$. So D_i is contained in a θ-fiber, but not in a θ'-fiber, and consequently $I_{\theta'} < I_\theta$.

$\theta^{-1}(\eta)$ contains curves as irreducible components: In this particular case, we have a divisor $D_{i'}$ and a curve $C_{j'}$ such that $D_{i'} \cap C_{j'} \neq \emptyset$.

Recall that all the fibers of the map $\iota_X|_{E_{j'}} : E_{j'} \to C_{j'}$ are mapped surjectively onto $\iota_Z^{-1}(\eta)$ and note that $\tilde{D}_i$, the strict transform of D_i, contains a fiber of ι_X. Because of that, $\tilde{D}_i$ is not contained in a θ'-fiber. We note in complete similarity to our argumentation above that a divisor is θ'-exceptional if and only if it is of type E_j or a strict transform of one of the D_i. Because none of the E_j is contained in a θ'-fiber, we obtain $I_{\theta'} < I_\theta$ as above.

□

PROPOSITION 7.10. *Let X, Y and Z be as spelled out at the beginning of this section and assume additionally that $\theta^{-1}(z)$ does not contain a divisor for all $z \in Z$. Then the map ϕ factors into a sequence of blow-downs.*

PROOF. Let $\psi : X \to W$ be a relative MORI contraction over Z. Since X is smooth, ψ is not a small contraction. Let $\rho : W \to Z$ be the canonically defined intermediate map.

First, we discuss the case where ψ is of fiber type. We claim that then $X \cong Y$, so that we are finished. The base variety W is a surface. Because θ-fibers do not contain divisors, we obtain $W \cong Z$. So X is already a $\mathbb{P}_1$-bundle over Z, hence $X \cong Y$.

Secondly, we consider the case that ψ is divisorial. We claim that there exists a relative contraction $\psi' : X' \to W'$ over Y! Let D be the ψ-exceptional divisor. D is not contained in a θ-fiber so ψ maps D to a curve. The subvariety $\theta^{-1}\theta(D)$ is not irreducible, because if it was, then ρ would have 0-dimensional fibers over $\theta(D)$, contradicting generic fiber dimension 1. Let E be one of the components of $\theta^{-1}\theta(D)$ where $\phi(E)$ is a curve. Lemma 7.1 guarantees the existence of a relative contraction over Y. Again, ψ' is either a fibration and $X \cong Y$ or divisorial, hence a blow-down. In the latter case, $W' \to Y \to Z$ fulfills all the assumptions of this theorem and we may start anew. □

REMARK 7.11. If Y is a the compactification of a line bundle, then the Y', as given in the last two propositions, is still a compactified line bundle. The reason is that there is a map $Y' \to Y$. Note that if E_1 and E_2 are disjoint sections in Y. then their preimages are disjoint sections in Y'.

3. The main result

THEOREM 7.12. *Let X be a smooth projective variety of dimension 3 which is almost homogeneous with respect to the algebraic action of a linear algebraic group G. Then either $G \cong SL_2$, and X is a compactification of SL_2/Γ, where $\Gamma < SL_2$ is a finite subgroup, or after equivariantly resolving the singularities of X, a sequence of blowing up followed by a sequence of blowing down, we obtain a variety X' which is one of the following:*

1. $\mathbb{P}_3$
2. *Q_3, the 3-dimensional quadric*
3. *a linear $\mathbb{P}_1$-bundle over a surface*
4. *a linear $\mathbb{P}_2$-bundle over $\mathbb{P}_1$.*

If G is solvable and we are in case (3), then we can take X' to be the compactification of a line bundle.

PROOF. By virtue of propositions 6.1, 6.2 and lemma 6.3, we know that either $G \cong SL_2$ or X is homogeneous and isomorphic to $\mathbb{P}_3$ or Q_3, in which cases we are finished, or there exists an equivariant rational map $X \dashrightarrow^{eq} Z$, where $Z \cong \mathbb{P}_3$ or $0 < \dim Z < 3$. After blowing up X, if necessary, we assume that this map is in fact regular.

If $Z \cong \mathbb{P}_3$, we know by proposition 7.3 that either we can blow down X to $\mathbb{P}_3$, or we can continue in the case that $\dim Z = 2$.

If Z is of dimension 1, i.e. $Z \cong \mathbb{P}_1$, then by lemma 7.4, we can assume that there exists a morphism $\phi : X \to Y$, where Y is either the minimal quadric bundle described in example 4.12 or a linear $\mathbb{P}_2$ bundle over Z. If Y is the quadric bundle, then by lemma 4.14 we can continue in the case where $Z \cong \mathbb{P}_3$. By proposition 7.7, assume that Y is fixpoint-free, or continue in the case $Z \cong \mathbb{P}_3$ or $\dim Z = 2$. If ϕ exists, then lemma 7.2 yields that ϕ factors into a sequence of blow-downs.

If $\dim Z = 2$, we perform a relative minimal model program over Z, ending with a $\mathbb{P}_1$-bundle over a surface. Without loss of generality, using lemma 7.9 and proposition 7.10, we can assume that Y is a bundle over Z and that the rational map

$X \dashrightarrow Y$ is a sequence of blow-downs. If G is solvable, then the proposition 5.32 allows us to choose Y as the compactification of a line bundle. □

REMARK 7.13. There exists a combinatorial classification for the compactifications of SL_2/Γ in [**MJ87**]. It should be possible to determine the minimal models.

4. Concluding Remarks

If G is solvable, the set of equivariant-birational models of G is extremely large since we can blow up any curve in a fixed divisor. Our result is optimal in the solvable case.

If G contains a nontrivial semisimple part S, there is by far less freedom. Slice theorems and linearization give a good description of the G-stable varieties, so that a complete (combinatorial) classification will certainly be possible. For example, if G has no solvable part, then we have

THEOREM 7.14. *Let X be a smooth projective variety which is almost homogeneous with respect to an algebraic action of a semisimple linear group G. Then X is one of the following:*

1. *$\mathbb{P}_3$, or a product of lower dimensional projective spaces*
2. *Q_3, the 3-dimensional quadric*
3. *F_{12}, the full flag variety*
4. *a linear $\mathbb{P}_1$-bundle over $\mathbb{P}_1 \times \mathbb{P}_1$, $G \cong SL_2 \times SL_2$, acting transitively on the base, or X can be obtained from this variety by blowing one of the at most two G-exceptional divisors down to a curve.*
5. *the compactification of SL_2/Γ, where $\Gamma < SL_2$ is a discrete subgroup.*

PROOF. We know by lemma 6.3 on page 58 that either $X \cong \mathbb{P}_3$ or Q_3, or there exists an equivariant rational map $X \dashrightarrow Z$, where Z is a G-homogeneous curve or surface. We blow up X in order to make this map regular and then carry out a relative MORI program until we obtain $X \dashrightarrow^{eq} Y \to Z$, where Y has a bundle structure over Z.

If Y is a $\mathbb{P}_2$-bundle over $\mathbb{P}_1$, then either there exists a G-stable section which can be blown up in order to obtain a $\mathbb{P}_1$-bundle, or there does not exist a G-stable section, implying that the ineffectivity of the G-action on $\mathbb{P}_1$ is either SL_2, acting via the irreducible 3-dimensional representation, or is SL_3. In any case, we know that the only matrices commuting with the representations are $\mathbb{C}^*.Id$, so that the bundle has to be trivial. Since there is no G-stable subset except for a unique divisor in the SL_2-case, and because this divisor cannot be blown down, the equivariant-birational model is again unique.

If Y is a $\mathbb{P}_1$-bundle, then Y is isomorphic either to $\mathbb{P}_2$, and $X \cong F_{12}$, the full flag variety, or $Y \cong \mathbb{P}_1 \times \mathbb{P}_1$. In the latter case, either the isotropy group of a point in Y contains SL_2, and $Y \cong \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ or is isomorphic to $B \times B$, where B is a BOREL group in SL_2, giving us a one or two G-exceptional divisors, depending on whether the action of the isotropy group has one or two fixed points in the fiber. Again, there is no equivariant birational model except for a possible blow-down of one of these divisors. □

REMARK 7.15. One could also produce this classification via the theory of spherical varieties.

If G is neither solvable nor semisimple, it should still be possible to give a list of the minimal models occurring in theorem 7.12 (i.e. those that are minimal in the sense that they admit a MORI contraction of fiber type) —we could, for instance, calculate all linear $\mathbb{P}_1$-bundles over S-almost homogeneous surfaces by means of determining the extension groups of sequences $0 \to \mathcal{O} \to \ldots \to \mathcal{O}(L) \to 0$, where $L \in \operatorname{Pic}(Y)$, and Y is a HIRZEBRUCH surface and discussing their S-module structure. Moreover, there are no divisors of G-fixed points so that we know exactly what can be blown up and down.

Another way to extend the results of this paper is to drop the assumption that G is a *linear* group. New varieties, which have non-trivial equivariant mappings to their ALBANESE tori, will occur. For such varieties, the map $X \to \operatorname{Alb}(X)$ factors into a sequence of blow-downs. It is then remaining to describe the minimal models: these will be linear $\mathbb{P}_1$- and $\mathbb{P}_2$- and very special quadric bundles.

Index

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