

BOUNDS FOR FAMILIES OF SINGULAR RATIONAL CURVES

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1. INTRODUCTION

Let X be a normal projective variety over the field \mathbb{C} of complex numbers and $H \subset \text{RatCurves}^n(X)$ a dominant family of rational curves such that for a general point $x \in X$ the subfamily $H_x \subset H$ of curves passing through x is proper. We recall [Kol96] that $\text{RatCurves}^n(X)$ is the normalization of the quasi-projective subset of $\text{Chow}(X)$ which parameterizes irreducible and generically reduced rational curves. It follows from Mori's classical work [CKM88] that such a family exists e.g. if X is a Fano manifold.

In this setup we obtain the following diagram:

$$\begin{array}{ccc} U & \xrightarrow{\iota} & X \\ \pi \downarrow & & \\ H_x & & \end{array}$$

where $U \subset X \times H$ is the universal family. If $x \in X$ is a general point, we have the restricted diagram

$$\begin{array}{ccc} U_x & \xrightarrow{\iota_x} & \text{locus}(H_x) \subset X \\ \pi_x \downarrow & & \\ H & & \end{array}$$

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The topic of this paper is to outline a proof showing that there are only finitely many curves parametrized by H_x which are singular at x . As an immediate application we will give a new characterization of the projective space in terms of rational curves. This generalizes Kobayashi-Ochiai's theorem [KO73] to normal varieties. A detailed proof, containing all the tedious technicalities, can be found in [Keb00] and will be published elsewhere.

The present article summarizes a talk given at the Saitama conference for Algebraic Geometry held in 2000. The author expresses his thanks to Prof. F. Sakai for the invitation. The results were gained while the author was visiting RIMS in Kyoto, and the author would like to thank Prof. Y. Miyaoka who acted as a host during that period.

2. BOUNDS FOR THE FAMILY OF SINGULAR CURVES ON X

Before stating the result it is convenient to introduce the following notation.

Notation 2.1. Call an irreducible curve $C \subset X$ “immersed” if the normalization morphism $\eta : \tilde{C} \rightarrow C$ has rank one everywhere. Let $H_x^{\text{Sing}} \subset H_x$ be the closed subfamily of singular curves and let $H_x^{\text{Sing,ni}} \subset H_x^{\text{Sing}}$ be the closed subfamily of non-immersed curves through x . Finally, let $H_x^{\text{Sing},x} \subset H_x^{\text{Sing}}$ be the closed subfamily of curves which have a singularity at x .

The following is the main result of this paper:

Theorem 2.2. *In the setup described in the introduction the following holds.*

1. *The parameter space $H_x^{\text{Sing},x}$ is at most finite.*
2. *If there exists a line bundle $L \in \text{Pic}(X)$ which intersects the curves with multiplicity 2, then $H_x^{\text{Sing},x}$ is empty.*

The remainder of section 2 will be concerned with a sketch of the proof of theorem 2.2. The proof is rather lengthy so that we subdivide it into several steps.

1. Assume that $H_x^{\text{Sing},x}$ is not empty because otherwise there is nothing to prove.
2. Replace the universal family U_x by a family where all fibers are singular plane cubics.
3. Show that each of the following assumptions yields a contradiction:
 - (a) All curves are immersed.
 - (b) No curve is immersed.
 - (c) There exists a line bundle of degree two.
4. Conclude.

2.1. Step (1) of the proof. A technical dimension count—which we are not going to detail in this outline—shows that

$$\dim H_x^{\text{Sing}} \geq \dim H_x^{\text{Sing},x} + 1.$$

Thus, the assumption implies that $\dim H_x^{\text{Sing}} \geq 1$, and we fix a proper 1-dimensional subfamily $H' \subset H_x^{\text{Sing}}$.

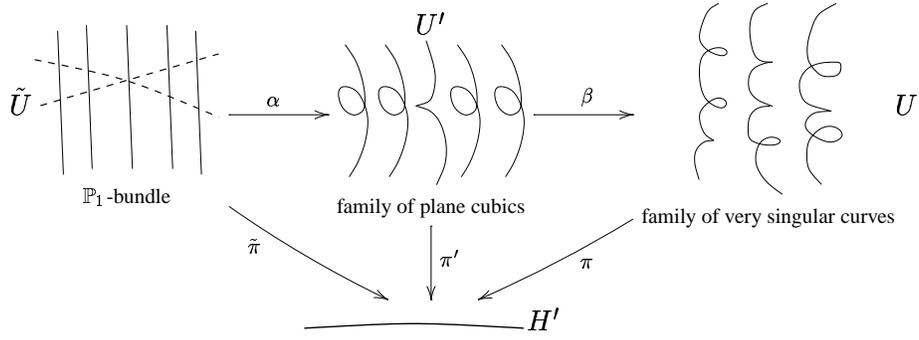


FIGURE 2.1. Replacing singular curves by plane cubics

2.2. **Step (2) of the proof.** Consider the diagram

$$(2.1) \quad \begin{array}{ccc} \tilde{U} & \xrightarrow[\text{normalization}]{\eta} & U \\ & \searrow \tilde{\pi} & \swarrow \pi \\ & & H' \end{array}$$

After performing a series of finite base changes, if necessary, we can assume that the following holds:

1. H' is smooth.
2. \tilde{U} is a \mathbb{P}_1 -bundle over H' . See [Kol96, thm. II.2.8] for this.
3. There exists a curve $s \subset U_{\text{Sing}}$ contained in the singular locus of U such that $\pi|_s$ is isomorphic. For this, let s be the normalization of a suitable component of U_{Sing} .
4. There exists a subscheme $\tilde{s} \subset \eta^{-1}(s)$ whose restriction to all $\tilde{\pi}$ -fibers is of length 2. For this, let \tilde{s} be the normalization of a curve in $\text{Hilb}_2(\eta^{-1}(s)/X)$ and note that the relative Hilb-functor commutes with base change. This is a bit technical, and we do not go into details here.

In this setup we would like to extend the diagram (2.1) to

$$\begin{array}{ccc} \tilde{U} & \xrightarrow[\text{normalization}]{\eta} & U \\ & \searrow \exists \alpha & \swarrow \exists \beta \\ & & U' \\ & \searrow \tilde{\pi} & \swarrow \pi \\ & & H \end{array}$$

$\exists \pi' \downarrow$

where all fibers of π' are rational curves with a single cusp or node, i.e. isomorphic to a plane cubic. Figure 2.1 depicts this setup. Here we explain only how to do this locally.

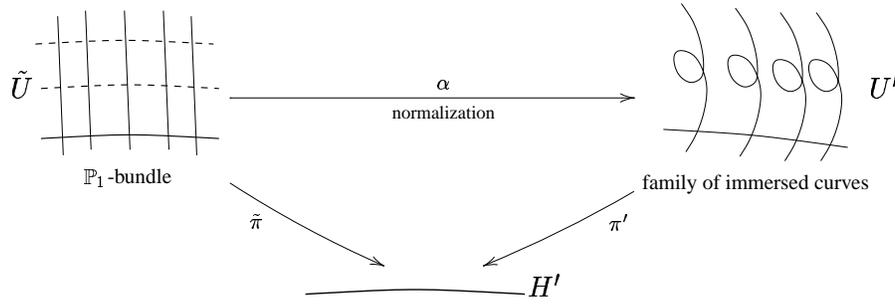


FIGURE 2.2. Family of immersed curves

Knowing that \tilde{U} is a \mathbb{P}_1 -bundle over H' , we find an (analytic) open set $V \subset H'$ with coordinate v , identify an open subset of $\tilde{\pi}^{-1}(V)$ with $V \times \mathbb{C}$, choose a bundle coordinate u and write

$$\tilde{s} = \{u^2 = f(v)\}$$

where f is a function on V . We would then define α to be

$$\begin{aligned} \alpha : V \times \mathbb{C} &\rightarrow V \times \mathbb{C}^2 \\ (v, u) &\mapsto (v, u^2 - f(v), u(u^2 - f(v))) \end{aligned}$$

A direct calculation shows that these locally defined morphisms glue together to give a global morphism $\alpha : \tilde{U} \rightarrow U'$, that a morphism $\beta : U' \rightarrow U$ exists and that the induced map $\pi' = \pi \circ \beta$ has the desired properties.

2.3. Step (3) of the proof. In order to conclude in the next step, we have to rule out several possibilities for the geometry of U' . We do this by reducing to the absurd, i.e. we assume that these settings exist and derive a contradiction.

2.3.1. The case where all curves are immersed. If all curves associated with H' are immersed, then the construction outlined above will automatically give a family U' where all fibers are isomorphic to nodal plane cubics —see figure 2.2. Let $\sigma_\infty \subset \tilde{U}$ be the section which is contracted to the point $x \in X$ (drawn as a solid line) and consider the preimage of the singular locus $\alpha^{-1}(U'_{\text{Sing}})$. After another finite base change, if necessary, we may assume that this set decomposes into two disjoint components $\alpha^{-1}(U'_{\text{Sing}}) = \sigma_0 \cup \sigma_1$, drawn as dashed lines. That way we obtain three sections σ_0 , σ_1 and σ_∞ in \tilde{U} , where σ_∞ can be contracted to a point and σ_0, σ_1 are disjoint. But then it follows from an elementary calculation with intersection numbers that either $\sigma_0 = \sigma_\infty$ or that $\sigma_1 = \sigma_\infty$. This however, is impossible, because then the Stein factorization of $\iota_x \circ \beta$ would contract both σ_0 and σ_1 , but ruled surfaces allow at most a single contractable section.

Remark 2.3. This setup has already been considered by several authors. See e.g. [CS95].

2.3.2. The case where no curve is immersed. If no curve associated with H' is immersed, then the curves s and \tilde{s} in the construction can be chosen so that U' is a family where all fibers are isomorphic

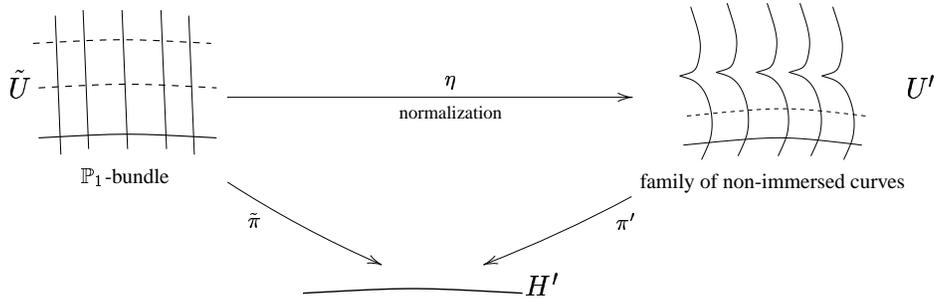


FIGURE 2.3. Family of immersed curves

to cuspidal plane cubics. See figure 2.3. Again we let $\sigma_\infty \subset \tilde{U}$ be the section which can be contracted and let σ_0 be the preimage of the singularities. That way we obtain two sections.

In order to obtain a third one, remark that if $C \subset \mathbb{P}_2$ is a cuspidal plane cubic and $H \subset \text{Pic}(C)$ a line bundle of positive degree $k > 0$, then there exists a unique point $y \in C_{\text{Reg}}$ such that $\mathcal{O}_C(ky) \cong H$. Thus, using the pull-back of an ample line bundle $L \in \beta^* \text{Pic}(X)$, we obtain a third section σ_1 which is disjoint from σ_0 . Now conclude as above. This time, however, it is not obvious that neither σ_0 nor σ_1 coincides with σ_∞ . Actually, this is true because $x \in X$ was chosen to be a general point. The proof for this is rather technical and can be found in [Keb00].

2.3.3. The case where a line bundle of degree 2 exists. In order to derive a contradiction to the assumption that a line bundle L of degree 2 exists, we argue along the same lines as above, noting that if $C \subset \mathbb{P}_2$ is a cuspidal nodal plane cubic and $H \subset \text{Pic}(C)$ a line bundle of degree 2, then there exist exactly two points $y_0, y_1 \in C_{\text{Reg}}$ such that $\mathcal{O}_C(2y_i) \cong H$. These will give two sections $\sigma_0, \sigma_1 \subset \tilde{U}$, and again we refer to [Keb00] for a proof that these are disjoint and do not coincide with σ_∞ .

2.4. Step (4) of the proof. If a line bundle $L \in \text{Pic}(X)$ of degree two exists, then it follows from the results of section 2.3.3 that $\dim H_x^{\text{Sing}} = 0$. But since the dimension count already mentioned gives

$$\dim H_x^{\text{Sing}} \geq \dim H_x^{\text{Sing},x} + 1,$$

we obtain that $\dim H_x^{\text{Sing},x} < 0$, i.e. $H_x^{\text{Sing},x} = \emptyset$. This proves assertion (2) of theorem 2.2.

In the other case we assume that $\dim H_x^{\text{Sing},x} \geq 1$, i.e. $\dim H_x^{\text{Sing}} \geq 2$ and one of the following must hold

1. H_x^{Sing} parameterizes only immersed curves
2. H_x^{Sing} parameterizes only non-immersed curves
3. H_x^{Sing} parameterizes both immersed and non-immersed curves

It turns out, however, that none of these possibilities occurs

1. Has been ruled out in section 2.3.1 above.
2. Has been ruled out in section 2.3.2 above.

3. It follows from an argument of Kollár that the closed subfamily of non-immersed curves is always of codimension at least one. Thus, the subfamily $H_x^{\text{Sing,ni}} \subset H_x^{\text{Sing}}$ is positive-dimensional, which has been ruled out in section 2.3.2.

This ends the proof of theorem 2.2.

3. CHARACTERIZATION OF PROJECTIVE SPACES

As an immediate application to theorem 2.2, we give a new characterization of the projective space.

Theorem 3.1. *Let X be a normal projective variety defined over \mathbb{C} , $L \in \text{Pic}(X)$ and $H \subset \text{RatCurves}^n(X)$ an irreducible component. Let $x \in \text{locus}(H)$ be a general closed point. Assume that H_x is proper and that $\text{locus}(H_x) = X$. If $L.C = 2$ for a curve $C \in H$, then $X \cong \mathbb{P}_n$.*

A similar result is given in [KS99] if X has at most \mathbb{Q} -factorial singularities. It follows from the classic results of Mori, when X is a Fano-manifold and $-K_X.C > \dim X$ for all rational curves $C \subset X$, then a family H exists where H_x is proper and $\text{locus}(H_x) = X$. Again we only outline a proof here and refer to [Keb00] for a detailed version.

Proof. Let \tilde{U}_x and \tilde{H}_x be the normalizations of the universal family U_x and the parameter space H_x , respectively. Consider the induced diagram

$$\begin{array}{ccc} \tilde{U}_x & \xrightarrow{\tilde{l}_x} & X \\ \tilde{\pi}_x \downarrow & & \\ \mathbb{P}_1\text{-bundle} & & \\ \tilde{H}_x & & \end{array}$$

By theorem 2.2, all curves associated with H_x are smooth at x . It then follows from an argumentation of Miyaoka (see e.g. [Kol96, V.3.7.5]) that \tilde{l}_x is birational. We claim that \tilde{H}_x is smooth; this follows from [Kol96, II.3] if X is a manifold, but also holds for normal varieties; we will, however, not give the proof here. In particular, it follows that \tilde{U}_x is smooth.

But then \tilde{l}_x must be isomorphic away from the section $\sigma_\infty := \tilde{l}_x^{-1}(x)$, which is contracted: otherwise, Zariski's main theorem asserts that there exists a point $x' \in X$ and a positive dimensional subfamily of curves passing through both x and x' . But Mori's bend-and-break argument (see e.g. [CKM88]) says that this cannot happen if H_x is proper. In particular, since \tilde{U}_x is smooth it follows that X is smooth.

In this setting, an elementary theorem of Mori yields the claim. See [Kol96, V.3.7.8]. □

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