# ASPECTS OF THE GEOMETRY OF VARIETIES WITH CANONICAL SINGULARITIES

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## 1. Introduction

This article is an extended version of two overview talks given by the authors in September 2013 at the Simons conference on "Foliation theory in algebraic geometry". We survey some recent developments regarding the global geometry of complex algebraic varieties with singularities occurring in the theory of minimal models. In other words, we are primarily interested in varieties with canonical or Kawamata log terminal (klt) singularities. More generally, we are interested in varieties $X$ carrying a $\mathbb{Q}$-divisor such that $(X, D)$ is klt, or perhaps log-canonical.

A typical problem is to understand the structure of complex projective varieties $X$ with numerically trivial canonical class $K_X$. These are the minimal models of manifolds $Y$ with Kodaira dimension $\kappa(Y) = 0$. If $X$ happens to be smooth, which is rare in minimal model theory, then powerful methods from analysis, such as existence results for Kähler-Einstein metrics, can be applied to study the structure of $X$. In the singular case however, new methods have to be developed, and we

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describe some steps along this way. For this, a good understanding of differential forms on varieties with canonical or klt singularities is required. Such an understanding is essential also in other circumstances, including moduli problems. Arguing along these lines, we show how the classical decomposition theorem for Kähler manifolds with vanishing first Chern class generalises to the singular case, to give a set of canonically defined foliations whose global geometry still needs to be explored.

In a similar vein, we recall a famous theorem of Yau, which asserts that any compact Kähler manifold with vanishing first and second Chern class is an étale quotient of a torus. Again, the proof of this result relies on the existence of a Kähler-Einstein metric. In case where $X$ is projective and has canonical singularities we will prove an analogous statement, where the quotient is étale in codimension one. In a certain sense, this statement can be seen as saying that the foliations constructed above are trivial, and that the second Chern class might be understood as an obstruction against their triviality. The proof requires a good understanding of the difference of the algebraic fundamental group of $X$ and that of its smooth locus. In particular, we need to understand the geometric meaning of the flatness of the smooth locus of $X$.

**Outline of the paper.** We give a short description of the content of the paper. As it might have become clear, there are two main technical tools: the theory of good (=reflexive) differentials and the study of fundamental groups. Part I is devoted to the study of differential forms, with the technical core, the Extension Theorem, described in Section 2 and the applications being given in the subsequent Sections 3–7. Part II first discusses algebraic fundamental groups of varieties with klt singularities in Section 8, followed in Sections 9 and 10 by applications to varieties whose regular part is flat and to varieties with trivial canonical and trivial second Chern class. Finally, we mention how the topological main result completes the structure theory of Nakayama-Zhang on polarised endomorphisms.

**Notation and global assumptions.** Throughout the paper, we work over the complex number field. We use standard notation and follow the conventions of minimal model theory, as introduced in [Har77, KM98]. We will frequently consider *quasi-étale* morphisms, a concept which might be non completely standard: A finite, surjective morphism of normal varieties $\gamma : X \to Y$ is called *quasi-étale* if $\gamma$ is étale in codimension one.

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This overview article summarises the content of the research articles [GKKP11, GKP11, GKP13a, GKP13b] as well as some recent developments by Graf, Jörder and others, and aims to put them into perspective. The results presented here are therefore not new. The exposition frequently follows the original articles. Proper references will be given throughout.

## Part I. Extension of differential forms and applications

### 2. Reflexive differentials and the Extension Theorem

2.1. **Statement of result.** We define the notion of reflexive differentials and state the main results of the paper [GKKP11] in this section. As there are other surveys available, we restrict ourselves to the minimal amount of material needed in later chapters and refer the reader to [Keb13a] for a more detailed introduction.

**Definition 2.1.** *Let $X$ be a normal variety or normal complex space. The sheaf of reflexive differentials on $X$ is defined to be*

$$\Omega_X^{[p]} := (\wedge^p \Omega_X^1)^{**},$$

*where $\Omega_X^1$ is the sheaf of Kähler differentials. If $D$ is a a reduced Weil divisor on $X$ and if $\Omega_X^1(\log D)$ denotes the sheaf of Kähler differentials with logarithmic poles along $D$, then*

$$\Omega_X^{[p]}(\log D) := (\wedge^p \Omega_X^1(\log D))^{**}.$$

*Notation* 2.2. Let $X$ be a normal variety or normal complex space. Given a coherent sheaf $\mathscr{A}$ on $X$ and a positive number $m$, set $\mathscr{A}^{[m]} := (\mathscr{A}^{\otimes m})^{**}$. If $f : X' \to X$ is any morphism, then $f^{[*]}(\mathscr{A}) := (f^*(\mathscr{A})^{**}$.

*Notation* 2.3. Let $X$ be a normal variety or normal complex space and $D$ a $\mathbb{Q}$-Weil divisor on $X$. A *log resolution of the pair* $(X, D)$ is a birational morphism $\pi : \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth, the exceptional locus $E$ has pure codimension one and the set $\pi^{-1}(\operatorname{supp} D) \cup E$ is a divisor with simple normal crossing support. By Hironaka's theorem, log resolutions always exist.

In a simplified form, the main result of [GKKP11] can be stated as follows.

**Theorem 2.4** ([GKKP11, Theorem 1.4])**.** *Let $X$ be a quasi-projective variety such that $(X, 0)$ is klt. Let $\pi$ be a log resolution of $(X, 0)$ and let $p$ be any number. Then $\pi_*\Omega_{\widetilde{X}}^p = \Omega_X^{[p]}$. Equivalently, $\pi_*\Omega_{\widetilde{X}}^p$ is reflexive.* □

In the most general version, we consider a log-canonical pair $(X, D)$. Then there exists a smallest closed algebraic set $N$ such that $(X, D)$ is klt outside $N$. The set $N$ is called the non-klt locus of $(X, D)$.

**Theorem 2.5** ([GKKP11, Theorem 1.5])**.** *Let $X$ be a quasi-projective variety carrying a $\mathbb{Q}$-Weil-divisor $D$ such that $(X, D)$ is log-canonical, with non-klt locus $N \subset X$. Let $\pi : \widetilde{X} \to X$ be a log resolution with exceptional set $E$, and let $\widetilde{D} \subset \widetilde{X}$ be the largest reduced divisor contained in $\pi^{-1}(N)$. Then $\pi_*\Omega_{\widetilde{X}}^p(\log \widetilde{D}) = \Omega_X^{[p]}(\log D)$, for all numbers $p$.* □

We refer to both theorems as "extension theorems". In fact, Theorem 2.5 can be restated as follows. Given any open set $U \subseteq X$ with preimage $\widetilde{U} = \pi^{-1}(U)$, Theorem 2.5 asserts that the restriction

$$\underbrace{H^0\big(\widetilde{U}, \Omega_{\widetilde{X}}^p(\log \widetilde{D})\big)}_{=H^0\left(U,\, \pi_*\Omega_{\widetilde{X}}^p(\log \widetilde{D})\right)} \to \underbrace{H^0\big(\widetilde{U} \setminus E, \Omega_{\widetilde{X}}^p(\log \widetilde{D})\big)}_{=H^0\left(U,\, \Omega_X^{[p]}(\log D)\right)}$$

is surjective, and hence isomorphic.

2.2. **Related results.** Building on results of Steenbrink-van Straten, [SvS85], Flenner proved in [Fle88] a version of Theorem 2.4 for $p \leq \operatorname{codim} X_{\text{sing}}$, for all normal varieties and without any assumption on the nature of the singularities. Namikawa showed Theorem 2.4 for $p \in \{1, 2\}$, provided that $X$ has canonical Gorenstein singularities, [Nam01, Section 1]. For further discussions we refer to [GKKP11].

We would like to emphasise that Theorems 2.4 and 2.5 are optimal if we want to have extension for all $p$. For examples and details, see [GKKP11, Section 3]. Relations to the notion of *Du Bois* singularities and pairs are discussed in [GK13, GK14]. Relations to $h$-differentials, the sheafification of Kähler differentials in Voevodsky's $h$-topology, are discussed in [JH13].

2.3. **Extension in the analytic category.** The Extension Theorems 2.4 and 2.5 are stated and proved in the algebraic category. In fact, the proof heavily uses parts of the minimal model program and certain vanishing theorems which are presently unavailable in the analytic category. However, there seems no reason why the Extension Theorem should not hold analytically. There is no problem to define notions as klt, log-canonical in the analytic category. In [GKP13a] a holomorphic version of Theorem 2.5 is established, provided the pair $(X, D)$ is locally algebraic. This is to say that every point $p \in X$ admits an open Euclidean neighbourhood $U$ which is open in a quasi-projective variety $Y$ such that $D|_U$ is the restriction of a divisor on $Y$. Following a famous algebraisation result of M. Artin, [Art69, Theorem 3.8], examples for locally algebraic spaces are provided by complex spaces with isolated singularities. Other examples are given by Moishezon spaces. The best known result in the analytic category reads as follows.

**Theorem 2.6** ([GKP13a, Section 2]). *Let $X$ be a normal complex locally algebraic variety carrying a $\mathbb{Q}$-divisor $D$ such that $(X, D)$ is log-canonical, with non-klt locus $N \subset X$. Let $\pi : \widetilde{X} \to X$ be a log resolution with exceptional set $E$. Let $\widetilde{D}$ be the largest reduced divisor contained in $\pi^{-1}(N)$. Then $\pi_* \Omega^p_{\widetilde{X}}(\log \widetilde{D})$ is reflexive for all $p$.* □

As pointed out, we expect that the Extension Theorems hold in full generality in the analytic category.

**Conjecture 2.7.** *Theorem 2.6 holds without the assumption that $X$ is locally algebraic.*

## 3. Kodaira-Akizuki-Nakano vanishing and the Poincaré lemma

3.1. **KAN type vanishing results for reflexive differentials.** Recall the statement of the classical Kodaira-Akizuki-Nakano Vanishing Theorem.

**Theorem 3.1** ([AN54]). *Let $X$ be a smooth projective variety and let $\mathscr{L}$ be an ample line bundle on $X$. Then*

$$H^q\big(X, \Omega^p_X \otimes \mathscr{L}\big) = 0 \quad \text{for } p + q > n, \text{ and} \tag{3.1.1}$$

$$H^q\big(X, \Omega^p_X \otimes \mathscr{L}^{-1}\big) = 0 \quad \text{for } p + q < n. \tag{3.1.2}$$ □

Assertions (3.1.1) and (3.1.2) are equivalent via Serre duality. Ramanujam [Ram72] gave a simplified proof of Theorem 3.1 and showed that it does not hold if one only requires $\mathscr{L}$ to be semi-ample and big. Esnault and Viehweg generalised Theorem 3.1 to logarithmic differentials, [EV86].

We ask for generalisations of Kodaira-Akizuki-Nakano vanishing to singular varieties, using reflexive differentials. In full generality, Kodaira-Akizuki-Nakano vanishing has been established for sheaves of reflexive differentials on varieties with quotient singularities, see [Ara88], as well as on toric varieties, see [CLS11, Theorem 9.3.1].

For varieties with more general types of singularities, vanishing results of KAN type are restricted to special values of $p$ and $q$. It turns out that even for spaces with isolated terminal Gorenstein singularities, Theorem 3.1 does not hold for arbitrary $p + q > n$, respectively $p + q < n$. We begin the discussion with one generalisation of Assertion (3.1.2).

**Theorem 3.2** ([GKP13a, Proposition 4.3]). *Let $X$ be a normal projective variety of dimension $n$, let $D$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, D)$ is log canonical, and let $\mathscr{L} \in \operatorname{Pic}(X)$ be an ample line bundle. Then*

$$H^0\big(X, \Omega^{[p]}_X(\log \lfloor D \rfloor) \otimes \mathscr{L}^{-1}\big) = 0 \qquad \text{for all } p < n, \text{ and} \tag{3.2.1}$$

$$H^1\big(X, \Omega^{[p]}_X(\log \lfloor D \rfloor) \otimes \mathscr{L}^{-1}\big) = 0 \qquad \text{for all } p < n - 1. \tag{3.2.2}$$

*If $(X, D)$ is additionally assumed to be dlt, then $H^q(X, \mathscr{L}^{-1}) = 0$ for all $q < n$.* □

The are analogous generalisations of Assertion (3.1.1).

**Theorem 3.3** ([GKP13a, Proposition 4.5]). *Let $X$ be a normal projective variety of dimension $n$, let $D$ be an effective $\mathbb{Q}$-divisor on $X$ such that $(X, D)$ is klt, and let $\mathscr{L} \in \operatorname{Pic}(X)$ be an ample line bundle. Then*

(3.3.1) $$H^q(X, \omega_X \otimes \mathscr{L}) = 0 \qquad \text{for all } q > 0, \text{ and}$$

(3.3.2) $$H^n(X, \Omega_X^{[p]} \otimes \mathscr{L}) = 0 \qquad \text{for all } p > 0. \quad \square$$

*Example* 3.4. The paper [GKP13a] exhibits a klt space $X$ of dimension 4 and an ample line bundle $\mathscr{L}$ such that

$$H^2(X, \Omega_X^{[1]} \otimes \mathscr{L}^{-1}) \neq 0 \quad \text{and} \quad H^2(X, \Omega_X^{[3]} \otimes \mathscr{L}) \neq 0.$$

It follows that Kodaira-Akizuki-Nakano does not hold in full generality on a klt space, even when the space has only Gorenstein, terminal singularities. The example given in [GKP13a] starts with the threefold $Y := \mathbb{P}(T_{\mathbb{P}_2})$. Set $\widetilde{X} = \mathbb{P}(\mathscr{O}_Y \oplus \mathscr{O}_Y(1))$ and let $\pi : \widetilde{X} \to X$ be the contraction of the divisor $E = \mathbb{P}(\mathscr{O}_Y))$. Then $p = \pi(E)$ is a terminal Gorenstein singularity. The calculations for the cohomology groups are lengthy. We refer the reader to [GKP13a] for details.

3.2. **Relation to the Poincaré lemma for reflexive differential forms.** Needless to say, the Poincaré lemma is fundamental in the theory of complex manifolds. It is therefore natural to ask to which extent it holds also for reflexive differentials on singular. Results in this direction have been obtained by several authors, including Campana-Flenner, Greuel and Reiffen. The singularities discussed in their work are often isolated, rational or holomorphically contractible. A rather complete list of references is found in Jörder's paper [Jö14]. For locally algebraic klt spaces, the Poincaré lemma holds in degree one.

**Theorem 3.5** ([GKP13a, Theorem 5.4]). *Let $X$ be a normal complex space and $D$ an effective $\mathbb{Q}$-divisor on $X$ such that $(X, D)$ is analytically klt and locally algebraic. Let $\sigma \in H^0(X, \Omega_X^{[1]})$ be a closed holomorphic reflexive one-form on $X$. Then every $p \in X$ has an open neighbourhood $U$ (in the Euclidean topology) and a holomorphic function $f \in \mathscr{O}_X(U)$ such that $\sigma|_U = df_U$.* □

The notion of *analytic klt spaces*, which is rather self-explaining, is properly introduced in [GKP13a]. Notice that it is not difficult to construct counterexamples to Theorem 3.5 if $(X, D)$ is only assumed to be log-canonical.

In his Freiburg Ph.D. thesis [Jö14] Jörder found a topological condition[1] which guarantees the validity of the Poincaré lemma in degree one, for normal, locally algebraic, complex spaces. Besides various other results, he showed that for projective varieties of dimension at least four with only one isolated rational singularity $p$, any failure of the Poincaré lemma in degree three yields

$$H^2(X, \Omega_X^{[q]} \otimes \mathscr{L}^{-1}) \neq 0, \quad \text{for any ample line bundle } \mathscr{L} \text{ over } X.$$

He also shows that the local divisor class groups of the singular points are obstructions to KAN type vanishing. In Example 3.4, it is the latter that give the non-vanishing, whereas the Poincaré lemma does hold everywhere. We refer the reader to [Jö14] for details. Poincaré lemmas in the context of $h$-differentials are also discussed there.

[1]vanishing of a local intersection cohomology group

## 4. Varieties with trivial canonical classes

4.1. **Decomposition of Kähler manifolds with vanishing first Chern class.** We recall the famous structure theorem for Kähler manifolds with vanishing first Chern class.

**Theorem 4.1** ([Bea83] and references there)**.** *Let $X$ be a compact Kähler manifold whose canonical divisor is numerically trivial. Then there exists a finite étale cover $X' \to X$ such that $X'$ decomposes as a product*

$$X' = T \times \prod_{\nu} X_\nu$$

*where $T$ is a compact complex torus, and where the $X_\nu$ are irreducible and simply-connected Calabi-Yau– or holomorphic-symplectic manifolds.* □

*Remark* 4.2. Let $X$ be a compact, simply-connected Kähler manifold. We call $X$ "Calabi-Yau" if $\omega_X \cong \mathscr{O}_X$ and $h^0(X, \Omega^p_X) = 0$ for all $p \notin \{0, \dim X\}$. We call $X$ "irreducible holomorphic-symplectic", if $\omega_X \cong \mathscr{O}_X$ and if there exists a non-degenerate two-form whose wedge powers generate the ring of differential forms.

4.2. **Decomposition of the tangent sheaf.** Important as it is, the class of manifolds with vanishing first Chern class is too small from the point of view of birational classification of projective (or compact Kähler) manifolds. There, we are generally more interested in the structure of manifolds $X$ with Kodaira dimension zero, $\kappa(X) = 0$. Conjecturally, any such $X$ possesses a good minimal model $X'$, which is $\mathbb{Q}$-factorial, has terminal singularities and a numerically trivial canonical divisor $K_{X'} \equiv_{\text{num}} 0$. Given one such $X'$, a theorem of Kawamata, [Kaw85b, Theorem 1.1], asserts that there exists a positive number $m$ such that $\mathscr{O}_X(mK_{X'}) \cong \mathscr{O}_{X'}$. We aim to prove a structure theorem for these varieties. Building on the Extension Theorem 2.4, the following infinitesimal analogue of Theorem 4.1 has been established in [GKP11].

**Theorem 4.3** ([GKP11, Theorem 1.3])**.** *Let $X$ be a normal, projective variety with at worst canonical singularities. Assume that the canonical divisor of $X$ is numerically trivial, $K_X \equiv 0$. Then, there exists an Abelian variety $A$, a projective variety $\widetilde{X}$ with at worst canonical singularities, a quasi-étale cover $f : A \times \widetilde{X} \to X$, and a decomposition*

$$\mathscr{T}_{\widetilde{X}} \cong \bigoplus \mathscr{E}_i$$

*such that the following holds.*

(4.3.1) *The $\mathscr{E}_i$ are integrable saturated subsheaves of $\mathscr{T}_{\widetilde{X}}$, with trivial determinants.*

*Further, if $g : \widehat{X} \to \widetilde{X}$ is any quasi-étale cover, then the following properties hold in addition.*

(4.3.2) *The sheaves $(g^*\mathscr{E}_i)^{**}$ are slope-stable with respect to any ample polarisation on $\widehat{X}$.*

(4.3.3) *The irregularity of $\widehat{X}$ is zero, $h^1(\widehat{X}, \mathscr{O}_{\widehat{X}}) = 0$.*

*Idea of proof.* Parts of the proof follow ideas of Bogomolov, [Bog74]. Consider a normal projective variety $X$ as in Theorem 4.3. Set $n := \dim X$. From the work of Kawamata [Kaw85a] we obtain a quasi-étale cover $f : A \times X' \to X$ where $A$ is an Abelian variety, where $\omega_{X'} \cong \mathscr{O}_{X'}$ and $q(X') = 0$, even after further quasi-étale covers of $X'$. Thus we will assume from now on that $q(X) = 0$, and we are allowed to pass to quasi-étale covers if we wish to do so.

Instead of decomposing $\mathscr{T}_X$ directly, we first show that there exists a decomposition in the ring of reflexive forms: given any number $p$ and any reflexive form $\sigma \in H^0(X, \Omega^{[p]}_X)$, we show that there exists a complementary form

$\tau \in H^0\big(X, \Omega_X^{[n-p]}\big)$ such that $\sigma|_{X_{\rm reg}} \wedge \tau|_{X_{\rm reg}}$ is a nowhere-vanishing top-form, defined on the smooth locus $X_{\rm reg}$. In other words, we show that the natural pairing given by the wedge product,

$$\bigwedge : H^0\big(X, \Omega_X^{[p]}\big) \times H^0\big(X, \Omega_X^{[n-p]}\big) \longrightarrow H^0\big(X, \omega_X\big) \cong \mathbb{C}, \tag{4.3.4}$$

is non-degenerate. For this, we express the Pairing (4.3.4) in terms of Dolbeault cohomology. The Extension Theorem 2.4 and the fact that canonical singularities are rational allows to compare the relevant cohomology groups with those that exist on a resolution $\widetilde{X}$ of singularities. Non-degeneracy of (4.3.4) comes out of non-degeneracy of the Serre duality pairings on $\widetilde{X}$.

In order to construct a decomposition of the tangent sheaf, recall from Miyaoka's work [Miy87a, Miy87b] that the tangent sheaf $\mathscr{T}_X$ is slope-semistable with respect to any polarisation. Assuming that there exists a polarisation $h$ where $\mathscr{T}_X$ is not stable, consider a destabilising subsheaf $\mathscr{E} \subsetneq \mathscr{T}_X$. It follows that the slope of $\mathscr{E}$ vanishes, $\mu_h(\mathscr{E}) = 0$, and it is easy to deduce from there that $c_1(\mathscr{E}) = 0$.

Passing to the *minimal dlt model*, we can assume that $X$ is $\mathbb{Q}$-factorial, [BCHM10]. Using $\mathbb{Q}$-factoriality and $q(X) = 0$, we conclude that $\det \mathscr{E} \cong \mathscr{O}_X$, perhaps after passing to another étale cover. If $r := \operatorname{rank} \mathscr{E}$, we obtain a subsheaf

$$\det \mathscr{E} \cong \mathscr{O}_X \subset \Omega_X^{[r]}.$$

In other words, we have constructed a reflexive differential form $\sigma \in H^0\big(X, \Omega_X^{[r]}\big)$. Using the existence of a complementary form $\tau \in H^0\big(X, \Omega_X^{[r]}\big)$, one can show by linear algebra that $\mathscr{E}$ is a direct summand of $\mathscr{T}_X$. □

The proof of Theorem 4.1 uses the existence of a Ricci-flat Kähler-Einstein metric quite heavily. In the singular setting, the necessary differential-geometric tools, namely a Kähler-Einstein metric on the smooth locus of $X$ with good boundary behaviour near the singularities, are not available so far —see [EGZ09] for recent developments in this direction. In order to pass from the infinitesimal decomposition of Theorem 4.3 to a physical decomposition of the variety as in Theorem 4.1, we would therefore propose to use different, more algebraic methods. The main problem is to show that the leaves of the foliation are algebraic, and then to analyse the structure of the closure of the leaves.

The following would be a conjectural analogue of Theorem 4.1. Together with the (conjectural) existence of good minimal models, a positive answer to this conjecture would give a rather satisfying structure theory for projective manifolds with vanishing Kodaira dimension.

**Problem 4.4.** *Let $X$ be a normal, $\mathbb{Q}$-factorial, projective variety with canonical singularities and trivial canonical class $K_X$. Suppose that $q(\widehat{X}) = 0$ for all quasi-étale covers $\widehat{X} \to X$. Then, there exists a quasi-étale cover $X' \to X$, such that $X'$ is birational to a product*

$$X' \sim_{\rm biratl} \prod_\nu X'_\nu,$$

*where the varieties $X'_\nu$ are $\mathbb{Q}$-factorial, with only canonical singularities, trivial canonical classes and the additional property that the tangent sheaf is* strongly stable*, that is, stable for any ample polarisation, even after passing to further quasi-étale covers.*

4.3. **Strongly stable varieties.** Whether or not Problem 4.4 has a positive solution, canonical varieties with linearly trivial canonical divisor and strongly stable tangent bundle will be important building blocks in any structure theory for varieties of Kodaira dimension zero. In analogy to the distinction between *irreducible* *complex-symplectic* and *Calabi-Yau* manifolds, one can distinguish the following two basic types.

**Definition 4.5.** *Let $X$ be a normal projective variety with $K_X \cong \mathscr{O}_X$, having at worst canonical singularities.*

(4.5.1) *We call $X$* Calabi-Yau *if $H^0\big(\widetilde{X},\, \Omega^{[q]}_{\widetilde{X}}\big) = 0$ for all numbers $0 < q < \dim X$ and all quasi-étale covers $\widetilde{X} \to X$.*

(4.5.2) *We call $X$* irreducible holomorphic-symplectic *if there exists a reflexive 2-form $\sigma \in H^0\big(X, \Omega^{[2]}_X\big)$ such that $\sigma$ is everywhere non-degenerate on $X_{\mathrm{reg}}$, and such that for all quasi-étale covers $f : \widetilde{X} \to X$, the exterior algebra of global reflexive forms is generated by $f^*(\sigma)$.*

We expect that the dichotomy known from the smooth case will also hold for singular varieties.

**Conjecture 4.6.** *Let $X$ be a projective variety with canonical singularities. If $\omega_X \cong \mathscr{O}_X$ and if $\mathscr{T}_X$ is strongly stable, then $X$ is either Calabi-Yau or irreducible holomorphic-symplectic, in the sense of Definition 4.5.*

*Remark* 4.7. The converse of Conjecture 4.6 is known to hold: The tangent sheaf of any Calabi-Yau or irreducible symplectic variety is strongly stable, [GKP11, Proposition 8.20].

In the smooth case, Calabi-Yau– and irreducible complex-symplectic manifolds are distinguished by their holonomy representation. As this depends again on the Ricci-flat Kähler metric, we cannot use holonomy in the singular setting. However, the following theorem does provide some evidence that the conjecture might in fact still be true.

**Theorem 4.8** ([GKP11, Propositions 8.15 and 8.21])**.** *Conjecture 4.6 holds if the dimension of $X$ is no more than five..* □

Theorem 4.8 has been shown using stability properties of wedge powers of $\mathscr{T}_X$. In fact, one way to attack Conjecture 4.6 is to observe that in the smooth case, the classes of Calabi-Yau and irreducible holomorphic-symplectic manifolds are distinguished by stability properties of $\bigwedge^2 \mathscr{T}_X$.

**Proposition 4.9.** *Let $X$ be a simply connected compact Kähler manifold with $c_1(X) = 0$. Fix an ample polarisation $h$. Then the following holds.*

(4.9.1) *The manifold $X$ is Calabi-Yau if and only if $\mathscr{T}_X$ and $\bigwedge^2 \mathscr{T}_X$ are both $h$-stable.*

(4.9.2) *The manifold $X$ is irreducible symplectic if and only $\mathscr{T}_X$ is $h$-stable and $\wedge^2 \mathscr{T}_X$ is $h$-semistable but not $h$-stable.*

*Idea of proof.* Using the Decomposition Theorem 4.1 and the smooth version of Theorem 4.3, we need only to show that the wedge power $\bigwedge^2 \mathscr{T}_X$ of a Calabi-Yau manifold $X$ is $h$-stable. If not, consider a destabilising subsheaf $\mathscr{S} \subset \bigwedge^2 \mathscr{T}_X$, say of rank $r$. Since $\det \mathscr{S} = \mathscr{O}_X$, we obtain the non-vanishing

$$H^0\big(X,\, \wedge^r \wedge^2 \mathscr{T}_X\big) \neq 0.$$

However —using holonomy and representation theory— it is a standard fact, although possibly never stated explicitly in the literature, that, with $n = \dim X$,

$$H^0\big(X,\, \mathscr{T}_X^{\otimes m}\big) = \begin{cases} 1 & \text{if } m \text{ is a multiple of } n \\ 0 & \text{otherwise.} \end{cases}$$

If $m$ is a multiple of $n$, the section comes from the direct summand $\mathscr{O}_X = (-aK_X) \subset \mathscr{T}_X^{\otimes m}$. This contradicts the above non-vanishing, since $\wedge^r \wedge^2 \mathscr{T}_X$ is a direct summand of some $\mathscr{T}_X^{\otimes m}$. □

To prove Conjecture 4.6 along these lines, a solution to the following problem would be needed.

**Problem 4.10** ([GKP11, Problem 8.11])**.** *Let $X$ be a normal projective variety of dimension $n > 1$ with $K_X \cong \mathscr{O}_X$, having at worst canonical singularities. Assume that the tangent sheaf $\mathscr{T}_X$ is strongly stable. Then show that the following holds.*

*(4.10.1) For any odd numbers $q \neq n$ and any quasi-étale cover $\widetilde{X} \to X$, we have $H^0\big(\widetilde{X}, \Omega^{[q]}_{\widetilde{X}}\big) = 0$.*

*(4.10.2) If there exists a quasi-étale cover $g : X' \to X$ and an even number $0 < q < n$ such that $H^0\big(X', \Omega^{[q]}_{X'}\big) \neq 0$, then there exists a reflexive 2-form $\sigma' \in H^0\big(X', \Omega^{[2]}_{X'}\big)$, symplectic on the smooth locus $X'_{\mathrm{reg}}$, such that for any quasi-étale cover $f : \widetilde{X} \to X'$, the exterior algebra of global reflexive forms on $\widetilde{X}$ is generated by $f^*(\sigma')$. In other words,*

$$\bigoplus_p H^0\Big(\widetilde{X}, \Omega^{[p]}_{\widetilde{X}}\Big) = \mathbb{C}\big[f^*(\sigma)\big].$$

It is shown in [GKP11, Proposition 8.21] that Problem 4.10 implies Conjecture 4.6. As indicated above, Problem 4.10 has been solved if $\dim X$ is at most five, [GKP11, Proposition 8.15]. It is certainly true if $X$ is smooth, [GKP11, Proposition 8.13]. We expect that in (4.10.2), it will be unnecessary to pass to the cover $\widetilde{X}$.

4.4. **The fundamental group.** The fundamental group $\pi_1(X)$ of a compact Kähler manifold $X$ with $c_1(X) = 0$ is almost Abelian. In other words, there exists an Abelian subgroup in $\pi_1(X)$ of finite index. The proof of this result does not require the Structure- and Decomposition Theorem 4.1, but nevertheless uses the existence of a Ricci-flat metric. A long-standing problem asks whether the same is true if only $\kappa(X) = 0$.

**Conjecture 4.11.** *Let $X$ be a projective (compact Kähler) manifold with $\kappa(X) = 0$. Then $\pi_1(X)$ is almost Abelian.*

If $X'$ is a minimal model of $X$, a result of Takayama [Tak03, Theorem 1.1], asserts that $\pi_1(X) \cong \pi_1(X')$. This leads us to conjecture the following.

**Conjecture 4.12.** *Let $X$ be a normal projective variety with at most terminal (canonical) singularities. If $K_X \equiv_{\mathrm{num}} 0$, then $\pi_1(X')$ is almost Abelian. If additionally $q(\widetilde{X}) = 0$ for any quasi-étale cover $\widetilde{X} \to X$, then $\pi_1(X)$ is finite.*

The following result in this direction has been established. The proof relies on Campana's work, [Cam95], and on the methods introduced in Section 4.2.

**Theorem 4.13** ([GKP11, Proposition 8.20])**.** *Let $X$ be a normal, $n$-dimensional, projective variety with at worst canonical singularities. If $K_X$ is torsion and if $\chi(X, \mathscr{O}_X) \neq 0$, then $\pi_1(X)$ is finite, of cardinality*

$$|\pi_1(X)| \leq \frac{2^{n-1}}{|\chi(X, \mathscr{O}_X)|}. \quad \square$$

**Theorem 4.14** ([GKP11, Corollary 8.25])**.** *Let $X$ be a normal projective variety with at worst canonical singularities. Assume that $\dim X \leq 4$, and that the canonical divisor $K_X$ is numerically trivial. Then $\pi_1(X)$ is almost Abelian, that is, $\pi_1(X)$ contains an Abelian subgroup of finite index.* $\square$

The case $n = 3$ has been shown previously in [Kol95, 4.17.3].

## 5. Rationally connected varieties

5.1. **Pluriforms on rationally connected varieties.** Rationally connected and rationally chain connected varieties play a prominent role in the structure theory of algebraic varieties. It is a basic fact that a rationally connected projective manifold $X$ does not carry any pluriform, that is

$$H^0\big(X, (\Omega^1_X)^{\otimes m}\big) = 0 \quad \forall m \in \mathbb{N}^+. \tag{5.0.1}$$

We refer the reader to [Kol96, IV.3.8] for a thorough discussion of this result. The key of the proof is the existence of many rational curves $C \subset X$ such that the restricted tangent bundle $T_X|_C$ is ample.

A well-known conjecture of Mumford asserts that (5.0.1) actually characterises rationally connected manifolds. This has been proven in dimension three by Kollár–Miyaoka–Mori, [KMM92, Thm. 3.2]. For an asymptotic version in any dimension, see [Pet06, CDP12]. As an immediate consequence of the Extension Theorem 2.4, the vanishing result (5.0.1) generalises to reflexive $p$-forms on spaces which support klt pairs.

**Theorem 5.1** ([GKKP11, Theorem 5.1]). *Let $X$ be a normal, rationally chain-connected projective variety. If there exists a $\mathbb{Q}$-divisor $D$ on $X$ such that $(X, D)$ is klt, then $H^0\big(X, \Omega^{[p]}_X\big) = 0$ for all $1 \le p \le \dim X$.* □

*Remark* 5.2. At this point, the following remark might be useful. Let $X$ be a normal, rationally chain-connected projective variety. If there exists a $\mathbb{Q}$-divisor $D$ on $X$ such that $(X, D)$ is klt, then $X$ is in fact rationally connected, cf. [HM07, Cor. 1.5].

It is natural to suspect that the vanishing (5.0.1) should also hold for pluriforms, that is, for section in reflexive tensor powers, $H^0\big(X, (\Omega^1_X)^{[m]}\big)$. Somewhat surprisingly, this is not always the case. This emphasises the fact that the statement of the Extension Theorem is not true for pluriforms. On the positive side, the following is known to hold.

**Theorem 5.3** ([GKP13a, Theorem 1.3]). *Let $X$ be a normal, rationally connected, projective variety. If $X$ is factorial and has canonical singularities, then*

$$H^0\left(X, (\Omega^1_X)^{[m]}\right) = 0 \quad \textit{for all } m \in \mathbb{N}^+, \textit{ where } (\Omega^1_X)^{[m]} := \big((\Omega^1_X)^{\otimes m}\big)^{**}. \quad \square$$

*Remark* 5.4 (Relation between Theorems 5.1 and 5.3). Let $X$ be a normal space. Assume that there exists a $\mathbb{Q}$-divisor $D$ on $X$ such that $(X, D)$ is klt. If $X$ is factorial, then $X$ has canonical singularities, cf. [KM98, Cor. 2.35].

*Remark* 5.5 (Necessity of the assumption that $X$ is canonical). There are examples of rational surfaces $X$ with log terminal singularities whose canonical bundle is torsion or even ample, cf. [Tot12, Example 10] or [Kol08, Example 43]. Since $H^0\big(X, \mathscr{O}_X(mK_X)\big) \subset H^0\big(X, (\Omega^1_X)^{[m\cdot\dim X]}\big)$, these examples show that the assumption that $X$ has *canonical* singularities cannot be omitted in Theorem 5.3.

The proof of Theorem 5.3 uses the notion of semistable sheaves on singular spaces, where semistability is defined respect to a movable curve class $\alpha$. We take the opportunity to correct an error in the proof given in [GKP13a]. An essential point in the proof is the fact that the reflexive tensor product of $\alpha$-semistable sheaves is again $\alpha$-semistable. In [GKP13a, Fact A.13], we referred to [CP11] for a proof, where however only the case that $\alpha$ is in the interior of the movable cone is established. The gap has been closed in [GKP14, Sect. 1.1.2 and Thm. 4.2].

Now, arguing by contradiction, one proceeds by analysing the maximal destabilising subsheaf $\mathscr{S}$ of a reflexive tensor power $(\Omega^1_X)^{[m]}$. The factoriality is used to conclude that $\det \mathscr{S}$ is a line bundle. This is important for calculations involving the restriction $\det \mathscr{S}|_C$: without the factoriality assumption, $\det \mathscr{S}|_C$ might contain torsion, which kills the argument. In fact, if $\mathscr{S}$ is a coherent sheaf on a smooth curve, then the positivity of $c_1(\mathscr{S})$ does not imply ampleness of $\mathscr{S}$. Instead, one might have $\mathscr{S} = A \oplus T$ with $A$ a negative line bundle and $T$ a (large) torsion sheaf.

*Remark* 5.6 (Theorem 5.3 in the $\mathbb{Q}$-factorial setting). If $X$ is not factorial in Theorem 5.3, but still $\mathbb{Q}$-factorial, then not only the proof of Theorem 5.3 fails, but the statement itself is false. A counterexample is given in [GKP13a, Example 3.7], by exhibiting a rationally connected surface $S$ such that $H^0(S, (\Omega^1_S)^{[2]}) \neq 0$.

Two recent preprints of Wenhao Ou, [Ou13, Ou14], describe the structure of rationally connected surfaces and threefolds with canonical singularities carrying a non-zero pluriform.

Following [Cam95] in the smooth case, a refined Kodaira dimension can be defined also in the singular case.

**Definition 5.7.** *Let $X$ be a normal, $\mathbb{Q}$-factorial, projective variety. Set*

$$\kappa^+(X) := \max\{\kappa(\det \mathscr{F}) \mid \mathscr{F} \subset \Omega^{[p]}_X \text{ a coherent subsheaf and } 1 \leq p \leq n\}.$$

Obviously, $\kappa^+(X) \geq \kappa(X)$. Unfortunately, $\kappa^+(X)$ does not behave well birationally, even when $X$ has canonical singularities. In fact, [GKP13a, Example 3.7] exhibits a rational surface $X$ supporting a rank-one, reflexive subsheaf $\mathscr{L} \subset \Omega^{[1]}_X$ such that $\mathscr{L}^{[2]}) = \mathscr{O}_X \subset (\Omega^1_X)^{[2]}$. Thus $\kappa^+(X) \geq 0$, whereas $\kappa^+(\widehat{X}) = -\infty$ for any desingularisation $\widehat{X}$ of $X$.

5.2. **The tangent bundle of rationally connected varieties.** As already mentioned, a rationally connected manifold $X$ carries many rational curves $C$ such that $\mathscr{T}_X|_C$ is ample. It is natural to ask whether this generalises to klt varieties: assume that $(X, \Delta)$ is klt or that $X$ has only canonical singularities. If $X$ is rationally (chain) connected, can one find rational curves $C$ through the general point of $X$ such that $\mathscr{T}_X|_C$ is ample[2]? The answer is negative in general.

**Proposition 5.8.** *Let $(X, \Delta)$ be klt and rationally connected. Suppose that $H^0(X, (\Omega^1_X)^{[m]}) \neq 0$ for some $m$. Then there is no irreducible curve $C$ through the general point of $X$, such that $\mathscr{T}_X|_C$ is ample. In particular, there does not exist a rational curve $C$ not meeting the singular locus of $X$ such that $\mathscr{T}_X|_C$ is ample.*

*Proof.* Fix a non-zero form $\omega \in H^0(X, (\Omega^1_X)^{[m]}) \neq 0$. Suppose to the contrary and assume that there is an irreducible curve $C$ through the general point $p$ of $X$, such that $\mathscr{T}_X|_C$ is ample. The form $\omega$ defines a morphism

$$\lambda : (\mathscr{T}_X^{\otimes m})^{**} =: \mathscr{T}_X^{[m]} \to \mathscr{O}_X.$$

Restricting to $C$ and observing that $C$ passes through a general point of $X$, we obtain a non-zero morphism

$$\lambda_C : \mathscr{T}_X^{[m]}|_C \to \mathscr{O}_C.$$

On the other hand, since $\mathscr{T}_X|_C$ is ample, so is $\mathscr{T}_X^{\otimes m}|_C = (\mathscr{T}_X^{\otimes m})|_C$. Using the generically injective map $\mathscr{T}_X^{\otimes m}|_C \to \mathscr{T}_X^{[m]}|_C$, we conclude that $\mathscr{T}_X^{[m]}|_C$ is ample. Hence $\lambda_C = 0$, a contradiction. □

[2] Observe that the sheaf $\mathscr{T}_X$ need not be locally free. We refer the reader to [Anc82, Section 2] for the definition of ampleness for arbitrary coherent sheaves.

5.3. **Related and complementary results.** In contrast to the non-existence of differential forms on rationally chain connected spaces non-existence of Kähler-differentials modulo torsion holds without any assumption as to the nature of the singularities.

**Theorem 5.9** ([Keb13b, Theorem 4.1]). *Let $X$ be a reduced, projective scheme. Assume that $X$ is rationally chain connected. Then $H^0\big(X,\, \Omega^p_X \big/ \mathrm{tor}\big) = 0$, for all $p$.* □

We do not assume that $X$ is irreducible. The statement of Theorem 5.9 becomes wrong if one replaces $\Omega^p_X \big/ \mathrm{tor}$ with Kähler differentials. Examples are given in [Keb13b, Section 4]. There are related results for $h$-differentials, [JH13].

## 6. The Lipman-Zariski Conjecture

The Lipman-Zariski Conjecture [Lip65, page 874], originally stated as a question, asserts that a normal variety $X$ whose tangent sheaf $\mathscr{T}_X$ is locally free, is smooth. Besides work of Lipman, the first results in this direction concern hypersurfaces and homogeneous complete intersections, and are due to Scheja-Storch [SS72, Chapter 9] and Hochster [Hoc75]. Generalising previous results by Steenbrink and van Straten, [SvS85], Flenner [Fle88] proved the Lipman-Zariski Conjecture if the singular locus of $X$ has codimension at least 3. Källström established the conjecture for complete intersections, [Kä11].

As a consequence of the Extension Theorem 2.4, we obtain the conjecture in the klt case, where the singular locus is of codimension two in general.

**Theorem 6.1** ([GKKP11, Theorem 6.1]). *Let $X$ be a normal, projective klt variety. In other words, assume that $(X, 0)$ is klt. If the tangent sheaf $\mathscr{T}_X$ is locally free, $X$ is smooth.*

*Idea of proof.* Like most other proofs of special cases of the Lipman-Zariski Conjecture, Theorem 6.1 is shown by lifting differential forms to a resolution of singularities. In our case, the Extension Theorem 2.4 allows to do that. We argue by contradiction and assume that $X$ is singular while $\mathscr{T}_X$ is locally free. Choose the so-called *functorial* or *canonical* resolution $\pi : \widetilde{X} \to X$, which is a log-resolution that commutes with smooth morphisms, see [Kol07]. By possibly shrinking $X$, we may assume that $\mathscr{T}_X$ is locally free; choose a basis $\theta_1, \dots, \theta_n$. These vector fields lift by [GKK10, Corollary 4.7] to logarithmic vector fields

$$\widetilde{\theta}_j \in H^0\big(\widetilde{X},\, \mathscr{T}_{\widetilde{X}}(-\log E)\big).$$

Choose the dual basis outside $E$ to obtain differential forms

$$\omega_j \in H^0\big(\widetilde{X} \setminus E,\, \Omega^1_{\widetilde{X}}\big), \quad \text{for all } 1 \le j \le n.$$

By the Extension Theorem 2.4, the $\omega_j$ are actually holomorphic forms on all of $\widetilde{X}$. The identity $\omega_i(\widetilde{\theta}_j) = \delta_{i,j}$ therefore holds everywhere on $\widetilde{X}$. However, the $\widetilde{\theta}_j$ are tangent to the exceptional divisor, providing a contradiction. □

**Further generalisations.** Recently, Graf, Graf-Kovács [GK13] and Druel [Dru13] generalised Theorem 6.1 to the log-canonical case. Druel's proof is independent of the Extension Theorem and instead uses foliation theory, while Graf-Kovács use an Extension Theorem for Du Bois pairs. Finally, we mention that Jörder proved the Lipman-Zariski Conjecture in case where $\mathscr{T}_X$ has a local basis of *commuting* vector fields [Jö13], and in case where there exists holomorphic $\mathbb{C}^*$ action with non-negative weights whose fixed point locus is not contained in the singular locus of $X$, [Jö14].

## 7. Bogomolov-Sommese vanishing and hyperbolicity of moduli spaces

The Extension Theorem 2.5 has been applied to prove hyperbolicity properties of moduli spaces. One of the further key ingredients is a generalisation of the Bogomolov-Sommese Vanishing Theorem to singular varieties. Since these matters are explained in quite some detail in the survey paper [Keb13a], we only recall the most important results here.

The most general version of the Bogomolov-Sommese vanishing is due to Graf [Gra13], generalising [GKKP11, Theorem 7.2]. We refrain from stating the most general form, which works in the context of "Campana orbifolds" or "$\mathcal{C}$-pairs", but just cite the following, more intuitive version.

**Theorem 7.1** ([Gra13, Theorem 1.3]). *Let $(X, D)$ be a normal, projective, log-canonical pair. Assume that $\mathscr{A} \subset \Omega_X^{[p]}(\log\lfloor D \rfloor)$ is a reflexive sheaf of rank 1. Then $\kappa(\mathscr{A}) \le p$.* □

It applies to moduli problems in the following way.

**Theorem 7.2** ([KK10, Corollary 1.3]). *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarised varieties, over a quasi-projective manifold $Y^\circ$ of dimension $\dim Y^\circ \le 3$. Then either*

*(7.2.1) $\kappa(Y^\circ) = -\infty$ and $\operatorname{Var}(f^\circ) < \dim Y^\circ$, or*
*(7.2.2) $\kappa(Y^\circ) \ge 0$ and $\operatorname{Var}(f^\circ) \le \kappa(Y^\circ)$.*

*Remark* 7.3. Recall that by definition, $\kappa(Y^\circ) = \kappa(K_Y + D)$, where $Y$ is a smooth projective compactification and $D = Y \setminus Y^\circ$.

*Idea of proof.* Consider the case where $Y := Y^\circ$ is projective and $K_Y$ linearly trivial. By Miyaoka's work [Miy87a, Miy87b], the sheaf of differential forms, $\Omega_Y^1$ will then be semistable with respect to any polarisation. Now, if there was a family $f^\circ : X^\circ \to Y^\circ$ of positive variation, it has been shown by Viehweg-Zuo that a suitable symmetric product of $\Omega_Y^1$ contains a positive subsheaf, violating semistablity.

If $Y^\circ$ is not projective, then it can be compactified to $Y$ by adding a boundary divisor $D$ with simple normal crossings. Assume for simplicity that $K_Y + D \equiv_{\text{num}} 0$ and that the Picard-Number of $Y_{\min}$ is one, so that any line bundle is either numerically trivial, ample or anti-ample. In this setting, Bogomolov-Sommese vanishing can be used to replace Miyaoka's semistability argument, which is not available in the presence of boundary divisors: by Viehweg-Zuo, the existence of a non-trivial family would imply that $\Omega_Y^1(\log D)$ is not semistable. However, any maximally destabilising subsheaf would automatically be ample, violating Bogomolov-Sommese.

If the simplifying assumptions are not satisfied and if $\dim Y^\circ \le 3$, then one can apply the minimal model program to come to a singular space $Y_{\min}$ with numerically trivial log-canonical class. With sufficient technical work, the extension theorem allows to work on these spaces, and to adopt the ideas sketched above. □

Theorem 7.2 is in fact a consequence of the following more general result.

**Theorem 7.4** ([JK11, Theorem 1.5]). *Let $f : X^\circ \to Y^\circ$ be a smooth family of canonically polarised varieties over a smooth quasi-projective base. If $Y^\circ$ is* special in the sense of Campana, *then the family $f$ is isotrivial.* □

*Remark* 7.5. In case where $Y^\circ$ is compact, a somewhat weaker version of Theorem 7.2 has been shown in all dimensions by Patakfalvi, [Pat12]. Generalisations of Theorems 7.2 and Theorems 7.4 to all dimensions are contained in a preprint by Campana-Păun, [CP13], and in the upcoming PhD thesis of Behrouz Taji.

# Part II. Local fundamental groups and étale covers

## 8. ÉTALE COVERS OF A KLT SPACE AND ITS SMOOTH LOCUS

8.1. **Finiteness of obstructions to extending finite étale covers from the smooth locus.** Working with a singular complex algebraic variety $X$, one is often interested in comparing the set of finite étale covers of $X$ with that of its smooth locus $X_{\mathrm{reg}}$. More precisely, one may ask the following.

**Question 8.1.** *What are the obstructions to extending finite étale covers of $X_{\mathrm{reg}}$ to $X$? How do the étale fundamental groups of $X$ and of its smooth locus differ?*

Our motivation to consider this question came from the study of varieties with canonical singularities and numerical trivial canonical classes and vanishing second Chern class in a suitable sense. This will be discussed in the subsequent Section 9.

*Remark* 8.2. If $X$ is normal, then it is a basic fact that the natural push-forward map between étale fundamental groups,

$$\widehat{\imath}_* : \widehat{\pi}_1(\widetilde{X}_{\mathrm{reg}}) \to \widehat{\pi}_1(\widetilde{X}), \tag{8.2.1}$$

is surjective. Question 8.1 therefore asks for conditions to guarantee injectivity.

Building on recent boundedness theorem of Hacon-McKernan-Xu for $\mathbb{Q}$-Fano klt pairs, [HMX12, Corollary 1.8], Chenyang Xu recently gave a complete answer for klt spaces with isolated singularities.

**Theorem 8.3** ([Xu12, Theorem 1])**.** *Let $0 \in (X, \Delta)$ be an analytic germ of an algebraic klt singularity. Then the algebraic local fundamental group $\widehat{\pi}_1^{\mathrm{loc}}(X, 0)$ is finite.* □

In the setting of Theorem 8.3, recall that $0 \in X$ admits a basis of neighbourhoods $U$ which are homeomorphic to the topological cone over the link $\mathrm{Link}(X, 0)$ The local fundamental group of $0 \in X$ is defined as the usual topological fundamental group of the link, that is, $\widehat{\pi}_1^{\mathrm{loc}}(X, 0) := \pi_1\big(\mathrm{Link}(X, s)\big)$. The algebraic local fundamental group is its profinite completion.

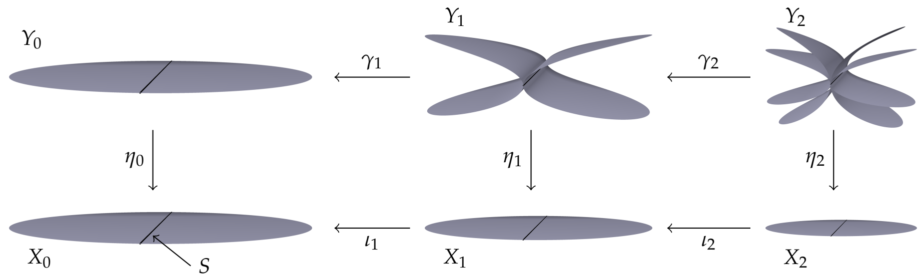
**Problem 8.4.** *It is an open question whether an analogue of Theorem 8.3 holds for the local fundamental group,* [Kol11, Question 26] *and* [Xu12, Conjecture 1]*.*

Building on Xu's result, the paper [GKP13b] establishes the following answer to Question 8.1. Recall from the Section 1 on page 2 that a finite surjective morphism $f : X \to Y$ between normal varieties is *quasi-étale*, if it is étale outside a set of codimension two. Equivalently, if $f$ is étale over the smooth locus of $Y$.

**Theorem 8.5** ([GKP13b, Theorem 1.4])**.** *Let $X$ be a normal, complex, quasi-projective variety. Assume that there exists a $\mathbb{Q}$-Weil divisor $\Delta$ such that $(X, \Delta)$ is klt. Then, there exists a normal variety $\widetilde{X}$ and a quasi-étale, Galois morphism $\gamma : \widetilde{X} \to X$, such that the following, equivalent conditions hold.*

- *(8.5.1) Any finite, étale cover of $\widetilde{X}_{\mathrm{reg}}$ extends to a finite, étale cover of $\widetilde{X}$.*
- *(8.5.2) The natural map $\widehat{\imath}_* : \widehat{\pi}_1(\widetilde{X}_{\mathrm{reg}}) \to \widehat{\pi}_1(\widetilde{X})$ of étale fundamental groups induced by the inclusion of the smooth locus, $\imath : \widetilde{X}_{\mathrm{reg}} \to \widetilde{X}$, is an isomorphism.* □

A few remarks and comments are perhaps in order. First of all, in Theorem 8.5 and throughout this paper, Galois morphisms are assumed to be finite and surjective, but need not be étale. Second, despite appearance to the contrary, Theorem 8.5 does *not* imply that the kernel of the push-forward morphism (8.2.1) is finite for all klt spaces. A counterexample is discussed in [GKP13b, Section 14.2]. Third, we point out that the variety $\widetilde{X}$ of Theorem 8.5 is not unique. In fact, it is shown




The figure shows the setup for the main result, Theorem 8.6, schematically. The morphisms $\eta_i$ are Galois covers over a sequence $X \supseteq X_0 \supseteq X_1 \supseteq \cdots$ is increasingly small open subsets of $X$. The morphisms $\gamma_i$ between these covering spaces are étale away from the preimages of $S$. In Theorem 8.6, the set $S$ is of codimension two or more. This aspect is difficult to illustrate and therefore not properly shown in the figure.

FIGURE 8.1. Setup of Theorem 8.6

in [GKP13b, Section 14.3] by way of example that a unique, minimal choice of $\widetilde{X}$ cannot exist in general.

8.2. **Generalisations.** Theorem 8.5 is in fact a corollary of the following, more general and much more involved result. In essence, Theorem 8.6 asserts that in any infinite tower of quasi-étale Galois morphisms over any sequence of increasingly smaller and smaller subsets of $X$, all but finitely many of the morphisms must in fact be étale.

**Theorem 8.6** ([GKP13b, Theorem 2.1]). *Let $X$ be a normal, complex, quasi-projective variety of dimension $\dim X \geq 2$. Assume that there exists a $\mathbb{Q}$-Weil divisor $\Delta$ such that $(X, \Delta)$ is klt. Suppose further that we are given a descending chain of dense open subsets $X \supseteq X_0 \supseteq X_1 \supseteq \cdots$, a closed reduced subscheme $S \subset X$ of codimension $\operatorname{codim}_X S \geq 2$, and a commutative diagram of morphisms between normal varieties,*

(8.6.1)
$$\begin{array}{ccccccccccc} & & Y_0 & \xleftarrow{\gamma_1} & Y_1 & \xleftarrow{\gamma_2} & Y_2 & \xleftarrow{\gamma_3} & Y_3 & \xleftarrow{\gamma_4} & \cdots \\ & & \downarrow{\scriptstyle \eta_0} & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \\ X & \xleftarrow[\iota_0]{} & X_0 & \xleftarrow[\iota_1]{} & X_1 & \xleftarrow[\iota_2]{} & X_2 & \xleftarrow[\iota_3]{} & X_3 & \xleftarrow[\iota_4]{} & \cdots, \end{array}$$

*where the following holds for all indices $i \in \mathbb{N}$.*

*(8.6.2) The morphisms $\iota_i$ are the inclusion maps.*
*(8.6.3) The morphisms $\gamma_i$ are quasi-finite, dominant and étale away from the reduced preimage set $S_i := \eta_i^{-1}(S)_{\mathrm{red}}$.*
*(8.6.4) The morphisms $\eta_i$ are finite, surjective, Galois, and étale away from $S_i$.*

*Then, all but finitely many of the morphisms $\gamma_i$ are étale. Further, if $S$ is not empty, then there exists an open subset $S^\circ \subseteq S$ and a number $N_S \in \mathbb{N}^+$, both depending only on $X$ and $S$, such that the following holds.*

*(8.6.5) Setting $S' := S \setminus S^\circ$, we have $\dim S' < \dim S$.*
*(8.6.6) Given any index $i \in \mathbb{N}$ and any point $y \in \eta_i^{-1}(S^\circ)$, the ramification index of $\eta_i^{an}$ at $y$ is bounded by $N_S$, that is, $r(\eta_i^{an}, y) < N_S$.* □

The setup of Theorem 8.6 is illustrated in Figure 8.1. To better understand its meaning and its relation to Theorem 8.5, it is useful to consider Theorem 8.6 in the special case where $X = X_0 = X_1 = X_2 = \cdots = Y_0$, where the morphisms $\gamma_i$ are finite and surjective, and where the morphisms $\eta_i$ are of the form

$$\eta_i = \begin{cases} \mathrm{Id}_X & \text{if } i = 0 \\ \gamma_1 \circ \cdots \circ \gamma_{i-1} \circ \gamma_i & \text{if } i > 0 \end{cases}$$

Under these assumptions, Theorem 8.6 reduces to the following.

**Theorem 8.7** ([GKP13b, Theorem 1.1]). *Let $X$ be a normal, complex, quasi-projective variety. Assume that there exists a $\mathbb{Q}$-Weil divisor $\Delta$ such that $(X, \Delta)$ is klt. Assume we are given a sequence of quasi-étale morphisms,*

(8.7.1) $$X = Y_0 \xleftarrow{\gamma_1} Y_1 \xleftarrow{\gamma_2} Y_2 \xleftarrow{\gamma_3} Y_3 \xleftarrow{\gamma_4} \cdots.$$

*If the composed morphisms $\gamma_1 \circ \cdots \circ \gamma_i : Y_i \to X$ are Galois for every $i \in \mathbb{N}^+$, then all but finitely many of the morphisms $\gamma_i$ are étale.* □

*Remark* 8.8. By purity of the branch locus, the assumption that all morphisms $\gamma_i$ of Theorem 8.7 are quasi-étale can also be formulated in one of the following, equivalent ways.

(8.8.1) All morphisms $\gamma_1 \circ \cdots \circ \gamma_i$ are étale over the smooth locus of $Y_0$.
(8.8.2) All morphisms $\gamma_i$ are étale over the smooth locus of $Y_{i-1}$.

Theorem 8.5 quickly follows from Theorem 8.7 by assuming to the contrary: if no cover of $X$ satisfied the conclusion of Theorem 8.5, we could inductively construct a tower of morphisms that are étale over the smooth loci, but not everywhere étale. Passing to the appropriate Galois closures, we can always achieve that the morphisms are Galois over $X$.

8.3. **Idea of proof.** The main idea for the proof of Theorem 8.6 is roughly formulated as follows. If $X$ has isolated singularities, then Theorem 8.6 can be easily deduced from Xu's work and nothing new needs to be done. If the singularities of $X$ are not isolated, we employ Verdier's topological triviality of algebraic morphisms, [Ver76], in order to construct a suitable Whitney stratification. Then argue inductively, stratum-by-stratum, using cutting-down-arguments to reduce to the case of isolated singularities.

8.4. **Immediate applications.** An array of morphisms as in Theorem 8.6 can inductively be constructed by fixing a point $p$ of a klt space $X$, by choosing Weil divisors $D_i \subset Y_i$ that are $\mathbb{Q}$-Cartier near the preimages of $p$ and taking the associated index-one-covers for the morphisms $\gamma_i$. The assertion that almost all morphisms $\gamma_i$ are étale then implies that the divisors in question were Cartier near the preimages of $p$. This way, one constructs a "simultaneous index-one cover" for all divisors that are $\mathbb{Q}$-Cartier in a neighbourhood of $p$.

**Theorem 8.9** ([GKP13b, Theorem 1.9]). *Let $X$ be a normal, complex, quasi-projective variety. Assume that there exists a $\mathbb{Q}$-Weil divisor $\Delta$ such that $(X, \Delta)$ is klt. Let $p \in X$ be any closed point. Then, there exists a Zariski-open neighbourhood $X^\circ$ of $p \in X$, a normal variety $\widetilde{X}^\circ$ and a quasi-étale, Galois morphism $\gamma : \widetilde{X}^\circ \to X^\circ$, such that the following holds for any Zariski-open neighbourhood $U = U(p) \subseteq X^\circ$ with preimage $\widetilde{U} = \gamma^{-1}(U)$.*

*(8.9.1) If $\widetilde{D}$ is any $\mathbb{Q}$-Cartier divisor on $\widetilde{U}$, then $\widetilde{D}$ is Cartier.*
*(8.9.2) If $D$ is any $\mathbb{Q}$-Cartier divisor on $U$, then $(\#\mathrm{Gal}(\gamma)) \cdot D$ is Cartier.* □

*Remark* 8.10 ([GKP13b, Remark 1.10]). Under the assumptions of Theorem 8.9, there exists a number $N \in \mathbb{N}^+$ such that $N \cdot D$ is Cartier, whenever $D$ is a $\mathbb{Q}$-Cartier divisor on $X$.

For applications on the global structure of Kähler space, as given below in the algebraic setting, it is highly desirable to extend the results presented in this section to the analytic category.

## 9. Flatness criteria and characterisation of torus quotients

9.1. **Extension results for flat sheaves.** We aim to apply Theorem 8.5 to the study of flat sheaves on klt spaces. Since we are dealing with singular spaces, we do not attempt to define flat sheaves via connections. Instead, a flat sheaf $\mathscr{F}$ will always be an analytic, locally free sheaf, given by a representation of the fundamental group. More precisely, we will use the following definition.

**Definition 9.1.** *If $Y$ is any complex space, and $\mathscr{G}$ is any locally free sheaf on $Y$, we call $\mathscr{G}$* flat *if it is defined by a representation of the topological fundamental group $\rho : \pi_1(Y) \to \mathrm{GL}_{\operatorname{rank} \mathscr{G}}(\mathbb{C})$. A locally free, algebraic sheaf on a complex algebraic variety $Y$ is called flat if and only if the associated analytic sheaf on the underlying complex space $Y^{an}$ is flat.*

Now consider a normal variety $X$ and a flat, locally free, analytic sheaf $\mathscr{F}^\circ$, defined on the complex manifold $X^{an}_{\mathrm{reg}}$. We aim to extend $\mathscr{F}^\circ$ across the singularities, to a coherent sheaf that is defined on all of $X$. Unlike in the algebraic case, where extension over subsets of codimension two is easy, the extension problem for coherent analytic sheaves is generally hard. For flat sheaves, however, a fundamental theorem of Deligne, [Del70, II.5, Corollary 5.8 and Theorem 5.9], asserts that $\mathscr{F}^\circ$ is algebraic, and thus extends to a coherent, algebraic sheaf $\mathscr{F}$ on $X$. If the algebraic fundamental groups of $X$ and $X_{\mathrm{reg}}$ agree, the following theorem shows that Deligne's extended sheaf $\mathscr{F}$ is again locally free and flat.

**Theorem 9.2** ([GKP13b, Section 11.1])**.** *Let $X$ be a normal, complex, quasi-projective variety, and assume that the natural inclusion map between étale fundamental groups, $\widehat{\iota}_* : \widehat{\pi}_1(\widetilde{X}_{\mathrm{reg}}) \to \widehat{\pi}_1(\widetilde{X})$, is isomorphic. If $\mathscr{F}^\circ$ is any flat, locally free, analytic sheaf defined on the complex manifold $X^{an}_{\mathrm{reg}}$, then there exists a flat, locally free, analytic sheaf $\mathscr{F}$ on $X^{an}$ such that $\mathscr{F}^\circ = \mathscr{F}|_{X^{an}_{\mathrm{reg}}}$.*

*Sketch of proof.* Set $Y := X^{an}$ and $Y^\circ := X^{an}_{\mathrm{reg}}$. The sheaf $\mathscr{F}^\circ$ then corresponds to a representation $\rho^\circ : \pi_1(Y^\circ) \to \mathrm{GL}(\operatorname{rank} \mathscr{F}, \mathbb{C})$. We need to show that this representation is induced by a representation of $\pi_1(Y)$. This is trivially true if the natural, surjective push-forward map of fundamental groups, $\iota_* : \pi_1(Y^\circ) \to \pi_1(Y)$ was known to be isomorphic. Our assumptions, however, guarantee only that the induced map $\widehat{\iota}_*$ between profinite completions is an isomorphism.

Write $G := \mathrm{img}(\rho^\circ)$. As a finitely generated subgroup of the general linear group, $G$ is residually finite by Malcev's theorem. Consequently, the profinite completion morphism $a : G \to \widehat{G}$ is injective. The remaining proof is now purely group-theoretic. □

A combination of Theorems 8.5 and 9.2 immediately gives the following consequence.

**Theorem 9.3** ([GKP13b, Theorem 1.13])**.** *Let $X$ be a normal, complex, quasi-projective variety. Assume that there exists a $\mathbb{Q}$-Weil divisor $\Delta$ such that $(X, \Delta)$ is klt. Then, there exists a normal variety $\widetilde{X}$ and a quasi-étale, Galois morphism $\gamma : \widetilde{X} \to X$, such that the following holds. If $\mathscr{G}^\circ$ is any flat, locally free, analytic sheaf on the complex space $\widetilde{X}^{an}_{\mathrm{reg}}$, there exists a flat, locally free, algebraic sheaf $\mathscr{G}$ on $\widetilde{X}$ such that $\mathscr{G}^\circ$ is isomorphic to the analytification of $\mathscr{G}|_{\widetilde{X}_{\mathrm{reg}}}$.* □

9.2. **Flatness criteria.** Theorem 9.3 can be used to show that many classical flatness criteria for semistable vector bundles, cf. [UY86, Kob87, Sim92, BS94], generalise to spaces with klt singularities, at least after passing to a suitable quasi-étale cover whose étale fundamental group coincides with that of its smooth locus.

9.2.1. *Chern classes on singular spaces.* In view of the applications, we are mostly interested in flatness criteria for semistable sheaves with vanishing first and second Chern classes. The literature discusses several competing notions of Chern classes on singular spaces, all of which are technically challenging, cf. [Mac74, Alu06]. We will restrict ourselves to the following elementary definition, which suffices in our case. We refer the reader to [GKP13b, Section 4] for more details.

**Definition 9.4.** *Let $X$ be a normal variety and $\mathscr{E}$ a coherent sheaf of $\mathscr{O}_X$-modules. A* resolution of $(X, \mathscr{E})$ *is a proper, birational and surjective morphism $\pi : \widetilde{X} \to X$ such that the space $\widetilde{X}$ is smooth, and such that the sheaf $\pi^*(\mathscr{E}) \big/ \text{tor}$ is locally free. If $\pi$ is isomorphic over the open set where $X$ is smooth and $\mathscr{E} / \text{tor}$ is locally free, we call $\pi$ a* strong resolution of $(X, \mathscr{E})$.

The existence of a resolution of singularities combined with a classical result of Rossi, [Ros68, Thm. 3.5], shows that resolutions and strong resolutions of $(X, \mathscr{E})$ exist.

**Definition 9.5.** *Let $X$ be a normal, $n$-dimensional, quasi-projective variety and $\mathscr{E}$ be a coherent sheaf of $\mathscr{O}_X$-modules. Assume we are given a number $i \in \mathbb{N}^+$ such that $X$ is smooth in codimension $i$ and such that $\mathscr{E}$ is locally free in codimension $i$. Given any resolution morphism $\pi : \widetilde{X} \to X$ of $(X, \mathscr{E})$ and any set of Cartier divisors $L_1, \dots L_{n-i}$ on $X$, we use the following shorthand notation*

$$c_i(\mathscr{E}) \cdot L_1 \cdots L_{n-i} := c_i(\mathscr{F}) \cdot (\pi^* L_1) \cdots (\pi^* L_{n-i}) \in \mathbb{Z}.$$

*where $\mathscr{F} := \pi^* \mathscr{E} \big/ \text{tor}$, and where $c_i(\mathscr{F})$ denote the classical Chern classes of the locally free sheaf $\mathscr{F}$ on the smooth variety $\widetilde{X}$.*

9.2.2. *Flatness criteria.* Using the above definitions, we generalize a famous flatness criterion of Simpson, [Sim92], to the klt setting.

**Theorem 9.6** ([GKP13b, Theorem 1.19])**.** *Let $X$ be an $n$-dimensional, normal, complex, projective variety, smooth in codimension two. Assume that there exists a $\mathbb{Q}$-Weil divisor $D$ such that $(X, D)$ is klt. Let $H$ be an ample Cartier divisor on $X$, and $\mathscr{E}$ be a reflexive, $H$-semistable sheaf. Assume that the following intersection numbers vanish*

$$c_1(\mathscr{E}) \cdot H^{n-1} = 0, \quad c_1(\mathscr{E})^2 \cdot H^{n-2} = 0, \quad \text{and} \quad c_2(\mathscr{E}) \cdot H^{n-2} = 0. \tag{9.6.1}$$

*Then, there exists a normal variety $\widetilde{X}$ and a quasi-étale, Galois morphism $\gamma : \widetilde{X} \to X$, such that $(\gamma^* \mathscr{E})^{**}$ is locally free and flat, that is, $(\gamma^* \mathscr{E})^{**}$ is given by a linear representation of $\pi_1(\widetilde{X})$.*

*Idea of proof.* The proof of Theorem 9.6 uses cutting-down arguments to reduce to the case of a smooth surface $S \subset X_{\text{reg}}$, where Simpson's flatness criterion [Sim92] can be applied. Hamm and Goreski-MacPherson's version of the Lefschetz theorem [GM88, II.1.2] implies that the sheaf $\mathscr{E}|_S$ extends to a flat sheaf that is defined on all of $X_{\text{reg}}$. Boundedness and some vanishing results for singular spaces identify this sheaf with $\mathscr{E}|_{X_{\text{reg}}}$. An application of Theorem 9.3 finishes the proof. □

9.3. **Characterisation of torus quotients.** As a classical consequence of Yau's theorem [Yau78] on the existence of Kähler-Einstein metrics, any Ricci-flat, compact Kähler manifold $X$ with vanishing second Chern class is covered by a complex torus, cf. [LB70, Thm. 12.4.3] and [Kob87, Ch. IV, Cor. 4.15]. Using the flatness criteria discussed above, we generalise this result to the singular case, when $X$ has terminal or canonical singularities.

To this end, recall from Theorem 6.1 that a klt space is smooth if and only if its tangent sheaf is locally free. Theorem 9.3 therefore implies the following criterion to guarantee that a given variety has quotient singularities and is a quotient of an Abelian variety.

**Theorem 9.7** ([GKP13b, Corollary 1.15]). *Let $X$ be a normal, complex, quasi-projective variety. Assume that $(X, D)$ is klt for some $\mathbb{Q}$-divisor $D$. If $\mathscr{T}_{X_{\mathrm{reg}}}$ is flat, then $\widetilde{X}$ is smooth and $X$ has only quotient singularities. If $X$ is additionally assumed to be projective, then there exists an Abelian variety $A$ and a quasi-étale Galois morphism $A \to \widetilde{X}$.* □

With sufficient amount of technical work, the flatness criterion of semistable sheaves with vanishing first and second Chern classes, Theorem 9.6, will then imply the following.

**Theorem 9.8** ([GKP13b, Theorem 1.16]). *Let $X$ be a normal, complex, projective variety of dimension $n$ with at worst canonical singularities. Assume that $X$ is smooth in codimension two and that the canonical divisor is numerically trivial, $K_X \equiv 0$. Further, assume that there exist ample divisors $H_1, \ldots, H_{n-2}$ on $X$ and a desingularisation $\pi : \widetilde{X} \to X$ such that $c_2(\mathscr{T}_{\widetilde{X}}) \cdot \pi^*(H_1) \cdots \pi^*(H_{n-2}) = 0$. Then, there exists an Abelian variety $A$ and a quasi-étale, Galois morphism $A \to X$.* □

There are, in fact, necessary and sufficient conditions for a variety to be a torus quotient, cf. [GKP13b, Section 12]. In dimension three, Theorem 9.8 has been established by Shepherd-Barron and Wilson in [SBW94], and our proof of Theorem 9.8 follows their line of reasoning. The article [SBW94] also asserts an variant of Theorem 9.8 for threefolds with canonical singularities.

## 10. Applications to endomorphisms of algebraic varieties

In this final section we discuss an application of Theorem 8.7 to polarised endomorphisms of algebraic varieties. First we provide the relevant definition.

**Definition 10.1.** *Let $X$ be a normal, complex, projective variety. An endomorphism $f : X \to X$ is called* polarised *if there exists an ample Cartier divisor $H$ and a positive number $q \in \mathbb{N}^+$ such that $f^*(H) \sim q \cdot H$.*

In [NZ10], Nakayama and Zhang study the structure of varieties admitting polarised endomorphisms. They conjecture in [NZ10, Conjecture 1.2] that any variety of this kind is either uniruled or covered by an Abelian variety, with a quasi-étale covering map. They prove the conjecture in [NZ10, Theorem 3.3] under an additional assumption concerning fundamental groups of smooth loci of Euclidean-open subsets of $X$, which turns out to be an immediate consequence of Theorem 8.7. The following result is thus established.

**Theorem 10.2** ([NZ10, Conjecture 1.2] and [GKP13b, Theorem 1.20]). *Let $X$ be a normal, complex, projective variety admitting a non-trivial polarised endomorphism. Assume that $X$ is not uniruled. Then, there exists an Abelian variety $A$ and quasi-étale morphism $A \to X$.* □

Theorem 10.2 has consequences for the structure theory of varieties with endomorphisms. The following results have been shown in [NZ10], conditional to the assumption that [NZ10, Conjecture 1.2] = Theorem 10.2 holds true. The definition of the invariant $q^\sharp$ is recalled below.

**Theorem 10.3** ([NZ10, Theorem 1.3] and [GKP13b, Theorem 13.1])**.** *Let $f : X \to X$ be a non-isomorphic, polarised endomorphism of a normal, complex, projective variety $X$ of dimension $n$. Then $\kappa(X) \leq 0$ and $q^\sharp(X, f) \leq n$. Furthermore, there exists an Abelian variety $A$ of dimension $\dim A = q^\sharp(X, f)$ and a commutative diagram of normal, projective varieties,*

$$
\begin{array}{ccccccc}
A & \xleftarrow{\ \omega\ } & Z & \xrightarrow{\ \rho\ } & V & \xrightarrow{\ \tau\ } & X \\
{\scriptstyle f_A}\downarrow & & {\scriptstyle f_Z}\downarrow & & {\scriptstyle f_V}\downarrow & & \downarrow{\scriptstyle f} \\
A & \xleftarrow[\text{flat, surjective}]{\ \omega\ } & Z & \xrightarrow[\text{biratl.}]{\ \rho\ } & V & \xrightarrow[\text{finite, surjective, quasi-étale}]{\ \tau\ } & X,
\end{array}
$$

*where all vertical arrows are polarised endomorphism, and every fibre of $\omega$ is irreducible, normal and rationally connected. In particular, $X$ is rationally connected if $q^\sharp(X, f) = 0$.*

*Moreover, the fundamental group $\pi_1(X)$ contains a finitely generated, Abelian subgroup of finite index whose rank is at most $2 \cdot q^\sharp(X, f)$.* □

*Remark* 10.4 ([NZ10, page 992f]). In the setting of Theorem 10.3, the number $q^\sharp(X, f)$ is defined as the supremum of irregularities $q(\widetilde{X}') = h^1(\widetilde{X}', \mathscr{O}_{\widetilde{X}'})$ of a smooth model $\widetilde{X}'$ of $X'$ for all quasi-étale morphism $\tau : X' \to X$ admitting an endomorphism $f' : X' \to X'$ with $\tau \circ f' = f \circ \tau$.

Even for general ramified endomorphisms $f : X \to X$ it is known that $X$ is uniruled. For this fact and further information we refer to [AKP08]. It would definitely be interesting to establish both theorems for polarised endomorphisms of Kähler varieties.

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